Asymptotic Behavior of a Critical Fluid Model for Bandwidth Sharing with General File Size Distributions*

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Abstract

This work concerns the asymptotic behavior of solutions to a critical fluid model for a data communication network, where file sizes are generally distributed and the network operates under a fair bandwidth sharing policy, chosen from the family of (weighted) α-fair policies introduced by Mo and Walrand [18]. Solutions of the fluid model are measure-valued functions of time. Under law of large numbers scaling, Gromoll and Williams [8] proved that these solutions approximate dynamic solutions of a flow level model for congestion control in data communication networks, introduced by Massoulié and Roberts [17].

In a recent work [6], we proved stability of the strictly subcritical version of this fluid model under mild assumptions. In the current work, we study the asymptotic behavior (as time goes to infinity) of solutions of the *critical* fluid model, in which the nominal load on each network resource is less than or equal to its capacity and at least one resource is fully loaded. For this we introduce a new Lyapunov function, inspired by the work of Kelly and Williams [14], Mulvany et al. [19] and Paganini et al. [20]. Using this, under moderate conditions on the file size distributions, we prove that critical fluid model solutions converge uniformly to the set of invariant states as time goes to infinity, when started in suitable relatively compact sets. We expect that this result will play a key role in developing a diffusion approximation for the critically loaded flow level model of Massoulié and Roberts [17]. Furthermore, the techniques developed here may be useful for studying other stochastic network models with resource sharing.

*Research supported in part by NSF grants DMS-1206772 and DMS-1712974, and the Charles Lee Powell Foundation. A preliminary form of some of the material in this paper was featured in the Le Cam Lecture delivered by RJW at the IMS Annual Meeting held in Vilnius, Lithuania, in July 2018.

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MSC2020 Mathematics Subject Classification: Primary 60F99, 60K30, 90B10; Secondary 60J25, 60K25, 90B18.

Keywords and phrases: Critical fluid model, α-fair bandwidth sharing, general file size distribution, Lyapunov function, asymptotic behavior, measure-valued fluid model, transport equation.

Contents

1 Introduction

The design, analysis and control of modern data communication networks such as the Internet present challenging problems, especially due to the heterogeneity, complexity and size of these networks. Mathematical models at various levels have been introduced in an effort to provide insight into these problems. In particular, Massoulié and Roberts [17] introduced a flow level model aimed at capturing the connection level stochastic dynamics of file arrivals and departures in a data communication network, where bandwidth is dynamically shared amongst flows that correspond to continuous transfers of individual elastic files. A natural family of "fair" policies for sharing bandwidth amongst all files was introduced by Mo and Walrand [18] around the same time. These policies are often referred to as (weighted) α-fair policies, since a parameter $\alpha \in (0, \infty)$ (and optional weight parameters) is associated with the family. The cases $\alpha = 1$ (proportional fairness) and $\alpha \to \infty$ (max-min fairness) have received particular attention.

Fluid models have proved useful for studying the stability of the flow level model operating under α-fair bandwidth sharing policies. These fluid models arise as limits, under law of large numbers scaling, from the original (stochastic) flow level model. Under general distributional assumptions on arrivals and file sizes, because bandwidth sharing is a generalization of processor sharing, it is convenient to use a measure-valued state descriptor for the flow level model that keeps track of residual file sizes. A result of Gromoll and Williams [8] establishes that, under law of large numbers scaling, a measure-valued process that tracks the dynamics of the flow level model operating under an α-fair policy, can be approximated by a measure-valued fluid model solution.

Studying the stability of fluid models for bandwidth sharing has been an active area of research since the publication of [17], with contributions from multiple authors including those of Bonald and Massoulié [1], De Veciana et al. [5], Kelly and Williams [14], Ye et al. [29], Chiang et al. [3], Massoulié [16] and Paganini et al. [20]. For more details up through 2019, see the introduction to [6]. Recently, in [6], under mild assumptions, we gave a complete proof of stability of strictly subcritical fluid models for the Massoulié-Roberts flow level model operating under a family of policies considered in [3], which slightly generalize the (weighted) α-fair policies of Mo and Walrand [18]. Our work [6] uses a modest generalization of a Lyapunov function introduced by Paganini et al. [20]; moreover, it does not need the strong smoothness assumptions on fluid model solutions assumed in [20], and it rigorously treats the realistic, but singular situation, where the fluid level on some routes becomes zero while other route levels remain positive. When coupled with the results of Lee [15], under suitable assumptions on the interarrival and file size distributions, this yields positive recurrence of an age-based Markovian state descriptor for the flow level model when the network is underloaded, i.e., the nominal load on each resource is strictly less than its capacity.

Beyond issues of stability, the performance of the flow level model when some resources are operating at or near capacity, is of particular interest. Indeed, as generally observed by Kelly and Laws [13], in the heavily loaded regime, important features of good control policies are often displayed in sharpest relief. Furthermore, system designers and managers often strive to position systems in this regime to achieve maximal utilization of resources. Diffusion approximations have provided useful and insightful measures of performance for various heavily loaded stochastic networks (see the survey article [27] and references therein). For open multiclass queueing networks, Bramson [2] and Williams [26] developed a modular approach to establishing diffusion approximations for these networks when heavily loaded. A key aspect of this approach was to analyze the asymptotic behavior of critical fluid model solutions and to use this analysis to establish a dimension reduction property called multiplicative state space collapse, which provided a crucial step in proving a diffusion approximation. (The fluid models associated with heavily loaded stochastic networks are called critically loaded, meaning that in the fluid model, the nominal load on each resource is less than or equal to its capacity and at least one resource is at capacity). Various authors have expanded and adapted the approach of Bramson and Williams, to establish diffusion approximations for a variety of other heavily loaded stochastic networks.

For the flow level model of Massoulié and Roberts [17], there are a few works establishing diffusion approximations under certain distributional, control or network assumptions. All of these use analysis of fluid models as a key ingredient. In general, it remains an open problem to establish a diffusion approximation for the flow level model with general interarrival time and file size distributions when operating under α-fair bandwidth sharing policies. We provide a brief summary of existing work in this area and then describe the main focus of this paper.

With Poisson arrivals and exponentially distributed file sizes, Kelly and Williams [14] studied the asymptotic behavior of a critical fluid model for the flow level model operating under an α-fair bandwidth sharing policy, and proved uniform convergence of fluid model solutions to an invariant manifold when starting in a compact set. Subsequently, Kang et al. [12] used this analysis to prove multiplicative state space collapse, and, in the case of proportional fair sharing ($\alpha = 1$), combined the result of [14] with an invariance principle for reflected Brownian motion [11], to prove a diffusion approximation for the heavily loaded flow level model under a mild local traffic condition. The latter condition was subsequently weakened to a full rank condition on the network structure by Ye and Yao [28].

The fluid model considered by Kelly and Williams [14] focused on the fluid limit of the flow count process; the latter is a Markovian process when arrivals are Poisson and file sizes are exponentially distributed. As noted above, for more generally distributed arrivals and file sizes, a larger state descriptor is usually needed. A special case of the flow level model is when there is a single type of file and a single resource or communication link. In this case, bandwidth sharing is the same as processor sharing, and a natural state descriptor is a measure on the positive half line that keeps track of residual file sizes (plus a variable that tracks residual interarrival times). The modular approach of Bramson and Williams has been adapted to this case. Specifically, a fluid model for a GI/GI/1 processor sharing queue was developed by Gromoll et al. [9], asymptotic analysis of the critical fluid model was carried out by Puha and Williams [22], and Gromoll [7] subsequently used this to prove state space collapse and a heavy traffic diffusion approximation for the processor sharing queue. For the full flow level model of Massoulié and Roberts [17], operating under the proportional fair sharing discipline ($\alpha = 1$ and with equal weights), when arrivals are given by Poisson processes and file sizes have a phase-type distribution, Vlasiou et al. [25] used a critical fluid model analysis to study the steady-state distribution of the flow count process.

In this paper, we analyze the asymptotic behavior (as time goes to infinity) of measure-valued solutions to the fluid model of Gromoll and Williams [8] for the α-fair bandwidth sharing policies of Mo and Walrand [18]. It is anticipated that this work will provide a crucial link in a modular approach to proving a diffusion approximation for the Massoulié-Roberts flow level model with general interarrival and file size distributions when operating under the aforementioned fair bandwidth sharing policies. The key to our analysis is a new Lyapunov function, the formulation of which was inspired by the work of Kelly and Williams [14], Mulvany et al. [19] and Paganini et al. [20]. Using this, under moderate conditions on the file size distributions for the fluid model, we prove that critical fluid model solutions converge uniformly to the set of invariant states (called the invariant manifold) as time goes to infinity, when started in suitable relatively compact sets.

The Lyapunov function, G, introduced here, is a non-negative function that involves the difference of two functions, H and $\underline{F}$. When H is applied to a measure-valued fluid model solution, it yields a function of time that is non-increasing and that is strictly decreasing when the fluid model solution is off the invariant manifold. The function $\underline{F}$ is a function of workload at the bottleneck resources, and when evaluated at a fluid model solution, it is non-decreasing with time. Consequently, when G is evaluated along a fluid model solution, it is strictly decreasing with time when the fluid model solution is off the invariant manifold. It is also zero when the fluid model solution is on the invariant manifold.

The structure of this paper is as follows. In Section 2, we recall the fluid model and the characterization of its invariant states as developed by Gromoll and Williams [8]. We also recall some preliminary properties of fluid model solutions, taken from [6]. In Section 3, we introduce key assumptions on fluid model parameters, under which our results will be proved. In particular, the file size distributions are assumed to be absolutely continuous (with respect to Lebesgue measure), to have finite first and second moments, and to have bounded hazard rate. In this section, we also define functions H, K and $\underline{F}$ that are used in defining our Lyapunov function G and proving its properties. We introduce $\mathcal{H}^\zeta$ (respectively $\mathcal{K}^\zeta$), the composition of H (respectively K) with a fluid model solution ζ. Under our assumptions, the function $\mathcal{K}^\zeta$ will be shown to be the density in time of $\mathcal{H}^\zeta$. This relationship between $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$, and a non-positive upper bound on $\mathcal{K}^\zeta$, is stated in the key result, Theorem 3.1, in Section 3.2.3. For the proof of this theorem, given in Section 6, we use a smooth approximation of fluid model solutions that was also used by Fu and Williams [6], and which is similar to a smoothing used by Puha and Williams [23] and Mulvany et al. [19]. For the proof of an associated lemma (Lemma 3.5) we also employ some inequalities (see Propositions 6.1–6.3) used by Fu and Williams [6], which are similar to ones developed by Paganini et al. [20]. Conditions for sharpness of an inequality in Lemma 3.5 are new here and useful. The function $\underline{F}$ is defined in Section 3.2.4 via an optimization problem, which is similar to one used by Kelly and Williams [14] for the case of Poisson arrivals and exponential file sizes.

Section 3 ends with a characterization of solutions of this optimization problem and of the optimization problem used to define the bandwidth sharing policy, and a further characterization of the invariant states for the fluid model. The proofs of these results are similar to those of results in [14]. Our Lyapunov function, G, and its composition, $\mathcal{G}^\zeta$, with a fluid model solution, ζ, is defined in Section 4. Key properties of G are stated there and proved in Section 7.3. In Section 5, we state the main results of this paper. These describe the asymptotic behavior of $\mathcal{G}^\zeta$ as time goes to infinity, i.e., that it decreases monotonically and converges uniformly to zero for all fluid model solutions starting in suitable relatively compact sets, and that fluid model solutions converge uniformly to the invariant manifold starting in such sets. The proofs of these main results are given in Section 8. These proofs draw on some arguments first introduced in [23], where the asymptotic behavior of a critical fluid model for a single class processor sharing queue was studied. These arguments were extended in [19] to a critical fluid model of a multiclass processor sharing queue. However, for the bandwidth sharing (network) model considered here, key details for many parts of the arguments are more complicated than in either of these prior works. In particular, our Lyapunov function is different, we have a much more general bandwidth allocation policy, and we need to deal with the singular, but realistic, situation where the fluid level for some routes reaches zero. In this work, in referencing arguments that we generalize from [23, 19], we shall generally refer to the first paper [23], from which the arguments were adapted for [19]. In the course of proving the main results, along the way, in Lemma 8.1 we prove that when there is non-zero fluid flow on a route, the ratio of the total fluid mass on the route to the bandwidth allocated to that route is bounded for all time, and we use this to prove in Lemma 8.2 that any fluid model solution starting in one of our relatively compact sets stays within a (larger) relatively compact set from the same family for all time, where our relatively compact sets are more general than those in [23]. Besides the proof of properties of G, Section 7 develops some properties of resource level workload, the relationship between H and $\underline{F}$, and a bound on the total mass of fluid model solutions when started in suitable relatively compact sets, as preliminaries to the proofs of the main results. For reference, the appendix gives some basic background on hazard rates.

1.1 Notation

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. For $x \in \mathbb{R}$, let $x^+ = \max(x, 0)$. Define $\mathbf{C}_b^1(\mathbb{R})$ (resp. $\mathbf{C}_b^1(\mathbb{R}_+)$) to be the set of once continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ (resp. $f : \mathbb{R}_+ \to \mathbb{R}$) that together with their first derivatives are continuous and bounded on $\mathbb{R}$ (resp. $\mathbb{R}_+$). Let $\mathbf{C}_c^\infty(\mathbb{R})$ be the set of infinitely differentiable functions defined on the real line that have compact support. Let $\mathbb{1}_A$ denote the indicator function of a set A and let $\mathbb{1} = \mathbb{1}_{\mathbb{R}_+}$.

Let $\mathbf{M}$ be the set of finite non-negative Borel measures on $\mathbb{R}_+$, endowed with the topology of weak convergence. If $\{\xi^n\}_{n=1}^\infty$ is a sequence in $\mathbf{M}$ converging (weakly) to $\xi \in \mathbf{M}$, we write $\xi^n \xrightarrow{w} \xi$ as $n \to \infty$. Given $\xi \in \mathbf{M}$, let $\mathbf{L}^1(\xi)$ denote the set of Borel measurable functions, defined from $\mathbb{R}_+$ into $\mathbb{R}$, that are integrable with respect to ξ. For $f \in \mathbf{L}^1(\xi)$, let $\langle f, \xi \rangle = \int_{\mathbb{R}_+} f \, d\xi$. Also for any non-negative Borel measurable function $f \notin \mathbf{L}^1(\xi)$, let $\langle f, \xi \rangle = +\infty$. For each $x \in \mathbb{R}_+$, let $\chi(x) = x$. Define $\mathbf{M}_1 = \{\xi \in \mathbf{M} : \langle \chi, \xi \rangle < \infty\}$. Let $\mathbf{K} = \{\xi \in \mathbf{M} : \xi(\{x\}) = 0 \text{ for all } x \in \mathbb{R}_+\}$, the set of continuous measures in $\mathbf{M}$, and let $\mathbf{A}$ denote the elements of $\mathbf{K}$ that are absolutely continuous (with respect to Lebesgue measure) on $\mathbb{R}_+$.

Let $\mathbb{N}$ denote the set of positive integers. For $\mathbf{I} \in \mathbb{N}$, let $\mathcal{I} = \{1, \ldots, \mathbf{I}\}$ and define

$$\mathbf{M}^\mathbf{I} = \{(\xi_1, \ldots, \xi_\mathbf{I}) : \xi_i \in \mathbf{M} \text{ for all } i \in \mathcal{I}\},$$

$$\mathbf{M}_1^\mathbf{I} = \{(\xi_1, \ldots, \xi_\mathbf{I}) : \xi_i \in \mathbf{M}_1 \text{ for all } i \in \mathcal{I}\},$$

$$\mathbf{K}^\mathbf{I} = \{(\xi_1, \ldots, \xi_\mathbf{I}) : \xi_i \in \mathbf{K} \text{ for all } i \in \mathcal{I}\},$$

$$\mathbf{A}^\mathbf{I} = \{(\xi_1, \ldots, \xi_\mathbf{I}) : \xi_i \in \mathbf{A} \text{ for all } i \in \mathcal{I}\}.$$

Here $\mathbf{M}^\mathbf{I}$ has its product topology and the other sets have the induced topologies as subsets of $\mathbf{M}^\mathbf{I}$. Fluid model solutions will take values in $\mathbf{M}^\mathbf{I}$ and we shall refer to the measure $\xi \in \mathbf{M}^\mathbf{I}$ that has ξ_i equal to the zero measure on $\mathbb{R}_+$ for all $i \in \mathcal{I}$, as the zero measure (in $\mathbf{M}^\mathbf{I}$) or the zero state (for the fluid model). Given a real-valued Borel measurable function $f \geq 0$, for $\xi \in \mathbf{M}^\mathbf{I}$, define $\langle f, \xi \rangle = (\langle f, \xi_1 \rangle, \ldots, \langle f, \xi_\mathbf{I} \rangle)$.

With its topology of weak convergence, $\mathbf{M}$ is a Polish space [21], and a metric (called the Prohorov metric) which induces this topology and under which $\mathbf{M}$ is complete and separable is defined as follows. For a Borel set $B \subset \mathbb{R}_+$ and $\epsilon > 0$, define

$$B^\epsilon = \{y \in \mathbb{R}_+ : \inf_{x \in B} |x - y| < \epsilon\}.$$

For $\xi, \eta \in \mathbf{M}$, the Prohorov distance between ξ and η is defined by

$$\mathbf{d}(\xi, \eta) = \inf\{\epsilon > 0 : \xi(B) \leq \eta(B^\epsilon) + \epsilon \text{ and } \eta(B) \leq \xi(B^\epsilon) + \epsilon, \\ \text{for all closed sets } B \subset \mathbb{R}_+\}.$$

For $\xi, \eta \in \mathbf{M}^\mathbf{I}$, define

$$\mathbf{d_I}(\xi, \eta) = \max_{i \in \mathcal{I}} \mathbf{d}(\xi_i, \eta_i). \tag{1.1}$$

For any $\emptyset \neq \mathcal{B} \subset \mathbf{M}^\mathbf{I}$ and $\xi \in \mathbf{M}^\mathbf{I}$, define

$$\mathbf{d_I}(\xi, \mathcal{B}) = \inf_{\eta \in \mathcal{B}} \mathbf{d_I}(\xi, \eta).$$

2 Fluid Model

Here we recall the fluid model developed by Gromoll and Williams [8] as a functional law of large numbers approximation to the flow level model of Massoulié and Roberts [17], when operating under one of the (weighted) α-fair bandwidth sharing policies of Mo and Walrand [18]. Beyond the assumptions in Gromoll and Williams [8], we assume here that the incidence matrix R has full row rank and that the file size distributions have finite second as well as first moments.

2.1 Network Structure, Arrivals and File Sizes

For positive integers $\mathbf{I}$ and $\mathbf{J}$, consider finitely many resources (e.g., communication links) labelled by $j \in \mathcal{J} \equiv \{1, \ldots, \mathbf{J}\}$, and a finite set of routes labeled by $i \in \mathcal{I} \equiv \{1, \ldots, \mathbf{I}\}$. A route $i \in \mathcal{I}$ is simply a non-empty subset of $\mathcal{J}$ and is interpreted as the set of resources used by the route. Let R be the $\mathbf{J} \times \mathbf{I}$ incidence matrix satisfying $R_{ji} = 1$ if resource j is used by route i, and $R_{ji} = 0$ otherwise. We assume that R has full row rank $\mathbf{J}$. Each resource $j \in \mathcal{J}$ has a fixed (bandwidth) capacity $C_j > 0$.

Fix a vector $\nu = (\nu_1, \ldots, \nu_\mathbf{I})$ where $\nu_i > 0$ for each $i \in \mathcal{I}$, and a vector $\vartheta = (\vartheta_1, \ldots, \vartheta_\mathbf{I})$ where for each $i \in \mathcal{I}$, ϑ_i is a Borel probability measure on $\mathbb{R}_+$ that does not charge the origin and has finite first and second moments, i.e., $\vartheta_i(\{0\}) = 0, \langle \chi, \vartheta_i \rangle < \infty$ and $\langle \chi^2, \vartheta_i \rangle < \infty$. For $i \in \mathcal{I}$, the constant ν_i represents the mean arrival rate of files to route i and ϑ_i represents the distribution for the sizes of files arriving to route i. For each $i \in \mathcal{I}$, $\mu_i \equiv \frac{1}{\langle \chi, \vartheta_i \rangle}$ is the reciprocal of the mean of the distribution ϑ_i and $\rho_i \equiv \frac{\nu_i}{\mu_i}$ is interpreted as the *nominal load* (average bandwidth needed) on route i. For each $i \in \mathcal{I}$, let ϑ_i^e be the *excess lifetime distribution* associated with ϑ_i. The probability measure ϑ_i^e is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$ and has density

$$p_i^e(x) = \mu_i \langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle \quad \text{for all } x \in \mathbb{R}_+. \tag{2.1}$$

Since ϑ_i has finite second moment, ϑ_i^e has finite mean given by

$$\langle \chi, \vartheta_i^e \rangle = \frac{\mu_i}{2} \langle \chi^2, \vartheta_i \rangle. \tag{2.2}$$

For each $i \in \mathcal{I}$, we define $N_i(x) = \langle \mathbb{1}_{[0,x]}, \vartheta_i \rangle$, $\overline{N}_i(x) = 1 - N_i(x)$, $N_i^e(x) = \langle \mathbb{1}_{[0,x]}, \vartheta_i^e \rangle$, and $\overline{N}_i^e(x) = 1 - N_i^e(x)$ for each $x \in \mathbb{R}_+$. Note that $\mu_i^{-1} = \int_0^\infty \overline{N}_i(x) dx$ and $p_i^e(x) = \mu_i \overline{N}_i(x)$ for all $x \in \mathbb{R}_+$.

2.2 Bandwidth Sharing Policy

The bandwidth allocations in the fluid model change dynamically as a function of the amount of fluid on each route. We consider the (weighted) α-fair policies of Mo and Walrand [18]. To specify these, we need the following notation. For each $z \in \mathbb{R}_+^{\mathbf{I}}$, let $\mathcal{I}_+(z) = \{i \in \mathcal{I} : z_i > 0\}$ and $\mathcal{O}(z) = \{\psi \in \mathbb{R}_+^{\mathbf{I}} : \psi_i = 0 \text{ for all } i \notin \mathcal{I}_+(z)\}$. Fix parameters $\alpha > 0$ and $\kappa_i > 0$ for each $i \in \mathcal{I}$. Let $\kappa = (\kappa_1, \ldots, \kappa_{\mathbf{I}})$. The following optimization problem is used to define the bandwidth sharing policy associated with the pair of parameters (α, κ). Given $z \in \mathbb{R}_+^{\mathbf{I}}$ (corresponding to an amount of fluid on each route in the fluid model), the vector $\phi(z)$ of bandwidth allocations associated with z is the unique value of $\psi \in \mathcal{O}(z)$ that solves the following *utility maximization problem:*

$$\text{maximize} \sum_{i \in \mathcal{I}_+(z)} \kappa_i z_i U\Big(\frac{\psi_i}{z_i}\Big) \text{ subject to } \sum_{i \in \mathcal{I}} R_{ji}\psi_i \le C_j \text{ for all } j \in \mathcal{J},\ \psi \in \mathcal{O}(z), \tag{2.3}$$

where $U : [0,\infty) \to [-\infty, \infty)$ is a utility function of the form

$$U(x) = \begin{cases} \frac{1}{1-\alpha} x^{1-\alpha} & \text{if } \alpha \neq 1, \\ \log(x) & \text{if } \alpha = 1. \end{cases}$$

For $i \in \mathcal{I}$, the quantity $\phi_i(z)$ is the bandwidth allocated to route i and $\phi_i(z)/z_i$ is the bandwidth allocated per unit of fluid on route i. Then, each unit of fluid on route i has utility $U(\phi_i(z)/z_i)$ and the utility for the total amount of fluid on route i is $z_i U(\phi_i(z)/z_i)$. Thus, the bandwidth allocation is chosen to maximize a weighted sum of the utilities of the amount of fluid on each route.

Remark 2.1. For $i \in \mathcal{I}_+(z)$, we have $\phi_i(z) > 0$ because $U(0) = -\infty$ if $\alpha \ge 1$, or $U(0) = 0$ and $U'(x) \to +\infty$ as $x \to 0$ if $\alpha \in (0,1)$. Let

$$\mathcal{S}(z) = \{\psi \in \mathbb{R}_+^{\mathbf{I}} : \psi_i > 0 \text{ for all } i \in \mathcal{I}_+(z), \psi_i = 0 \text{ for all } i \notin \mathcal{I}_+(z)\}.$$

Then one can restrict the choice of ψ to the set $\mathcal{S}(z)$ for the utility maximization problem. The uniqueness of the maximizer follows from the strict concavity of the utility function U. Furthermore, for each $z \in \mathbb{R}_+^{\mathbf{I}}$, $\phi_i(\cdot)$ is continuous at z for each $i \in \mathcal{I}_+(z)$. This was proved by Kelly and Williams [14].

Remark 2.2. A slight generalization of the above bandwidth sharing policy has been considered by some authors, where the utility function U is allowed to depend on $i \in \mathcal{I}$, by replacing α by $\alpha_i \in (0,\infty)$ in the above. Chiang et al. [3] showed that with this policy, the fluid model described below (with a zero initial condition) can be obtained from the Massoulié-Roberts flow level model via a large capacity, law of large numbers scaling limit; this is different from the fixed capacity limit considered in [8] but the fluid model is the same. The stability of the strictly subcritical fluid model under this slightly generalized policy was shown in [6]. For the critical fluid model studied in this paper, our proof of Theorem 5.1, which shows that our Lyapunov function decreases along fluid model solutions, extends to the situation where α_i depends on i. However, our proofs of Theorems 5.2 and 5.3, which demonstrate that the Lyapunov function decreases to zero and fluid model solutions converge to the invariant manifold, depend on the scaling property that $\phi_i(rz) = \phi_i(z)$ for all $i \in \mathcal{I}$, $z \in \mathbb{R}_+^{\mathbf{I}}$ and $r > 0$. This property does not hold when α_i depends on i.

2.3 Definition of Fluid Model Solutions

The fluid model of Gromoll and Williams [8], with the bandwidth sharing policy described in the previous subsection, is described below. For the remainder of the paper, the parameters $(R, C, \alpha, \kappa, \nu, \vartheta)$ are fixed and the bandwidth allocation function ϕ is as specified in the previous section.

Definition 2.1. *Given a continuous function $\zeta : [0,\infty) \to \mathbf{M}^{\mathbf{I}}$, define the auxiliary functions (z, Λ, τ, u, w)*

by the following for all $t \geq 0$:

$$\begin{aligned} z(t) &= \langle 1, \zeta(t) \rangle, \\ \Lambda(t) &= \phi(z(t)), \\ \tau_i(t) &= \int_0^t \Big(\Lambda_i(s) \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) + \rho_i \mathbb{1}_{\{0\}}\big(z_i(s)\big) \Big) ds, \quad i \in \mathcal{I}, \\ u(t) &= Ct - R\tau(t), \\ w(t) &= \langle \chi, \zeta(t) \rangle. \end{aligned}$$

In Definition 2.1, the i^{th} component of $w(\cdot)$ represents the fluid workload for route i, $w_i(t) = \langle \chi, \zeta_i(t) \rangle$, $t \geq 0$. Note that $w(t) \in [0, \infty]^{\mathbf{I}}$ for each $t \geq 0$.

A fluid model solution is defined through projections against test functions in the class

$$\mathcal{C} = \{ f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = f'(0) = 0 \}. \tag{2.4}$$

Definition 2.2. *A fluid model solution associated with the parameters $(R, C, \alpha, \kappa, \nu, \vartheta)$ is a continuous function $\zeta : [0, \infty) \to \mathbf{M}^{\mathbf{I}}$ that, together with its auxiliary functions (z, Λ, τ, u), satisfies:*

- *(i)* $\langle \mathbb{1}_{\{0\}}, \zeta(t) \rangle = 0$ *for all* $t \geq 0$.
- *(ii) the function u_j is non-decreasing for all $j \in \mathcal{J}$,*
- *(iii) for each $f \in \mathcal{C}$, $i \in \mathcal{I}$, and $t \geq 0$,*

$$\langle f, \zeta_i(t) \rangle = \langle f, \zeta_i(0) \rangle - \int_0^t \langle f', \zeta_i(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s)) ds + \nu_i \langle f, \vartheta_i \rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds. \tag{2.5}$$

Remark 2.3. The auxiliary function w associated with a fluid model solution ζ satisfies the following for all $t \geq 0$ for those i for which $w_i(0) < \infty$:

$$w_i(t) = w_i(0) + \int_0^t \Big(\rho_i - \Lambda_i(s) \Big) \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds. \tag{2.6}$$

See Lemma 3.3 of Gromoll and Williams [8] for the method of proof of this fact.

Remark 2.4. The third property in Definition 2.2 can be extended to hold for all functions $f \in \tilde{\mathcal{C}} = \{ f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = 0 \}$. A proof of this is given in Lemma A.2 of Fu and Williams [6].

Remark 2.5. The fluid limit result proved by Gromoll and Williams [8] yields fluid model solutions which have initial states that are continuous measures and which have finite workload, i.e., for which $\zeta(0) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}_1^{\mathbf{I}}$. Indeed, in order for fluid model solutions to be continuous functions of time, the initial condition cannot have any atoms. For the analysis of Fu and Williams [6], the initial condition requires $\zeta(0) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}_1^{\mathbf{I}}$. Here for our analysis of the critical case, we will ultimately assume that $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ for some $\upsilon > 0$, where

$$\mathbf{K}_\upsilon^{\mathbf{I}} = \{ \xi \in \mathbf{K}^{\mathbf{I}} : \langle \mathbb{1}_{[x,\infty)}, \xi_i \rangle \leq \upsilon \langle \mathbb{1}_{[x,\infty)}, \vartheta_i^e \rangle \text{ for all } x \in \mathbb{R}_+, i \in \mathcal{I} \}. \tag{2.7}$$

We note that since $\xi \in \mathbf{K}^{\mathbf{I}}$ and ϑ^e have no atoms, in (2.7), $\mathbb{1}_{[x,\infty)}$ can be replaced by $\mathbb{1}_{(x,\infty)}$ without changing the definition. So we can use the following alternative representation:

$$\mathbf{K}_\upsilon^{\mathbf{I}} = \{ \xi \in \mathbf{K}^{\mathbf{I}} : \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle \leq \upsilon \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle \text{ for all } x \in \mathbb{R}_+, i \in \mathcal{I} \}, \tag{2.8}$$

We shall define certain functions on

$$\mathbf{M}_\upsilon^{\mathbf{I}} = \{ \xi \in \mathbf{M}^{\mathbf{I}} : \langle \mathbb{1}_{[x,\infty)}, \xi_i \rangle \leq \upsilon \langle \mathbb{1}_{[x,\infty)}, \vartheta_i^e \rangle \text{ for all } x \in \mathbb{R}_+, i \in \mathcal{I} \}, \tag{2.9}$$

which contains the closure of $\mathbf{K}_\upsilon^{\mathbf{I}}$. Note that in (2.9), we cannot replace $\mathbb{1}_{[x,\infty)}$ by $\mathbb{1}_{(x,\infty)}$, without changing the definition. Indeed, if $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$, then $\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle \leq \upsilon \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$ for all $x \in \mathbb{R}_+$, $i \in \mathcal{I}$, but the converse is not true in general as ξ_i could have an atom at zero. Note that for any $\xi \in \mathbf{M}_v^{\mathbf{I}}$, we have for each $i \in \mathcal{I}$, $\langle \mathbb{1}, \xi_i \rangle \le v$ and

$$\langle \chi, \xi_i \rangle = \int_0^\infty \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle dx \le v \int_0^\infty \overline{N}_i^e(x) dx = v \langle \chi, \vartheta_i^e \rangle = \frac{v\mu_i}{2} \langle \chi^2, \vartheta_i \rangle < \infty. \tag{2.10}$$

It follows that $\mathbf{M}_v^{\mathbf{I}}$ is compact as a subset of $\mathbf{M}^{\mathbf{I}}$ and so $\mathbf{K}_v^{\mathbf{I}}$, although not closed, is relatively compact as a subset of $\mathbf{M}^{\mathbf{I}}$; see Lemma 15.7.5 of Kallenberg [10] for the method of proof.

A small comment on notation is in order here. In this paper, we only refer to $\mathbf{M}_v^{\mathbf{I}}$ with general $v > 0$. Consequently, when we refer to $\mathbf{M}_1^{\mathbf{I}}$, we do not mean $\mathbf{M}_v^{\mathbf{I}}$ with $v = 1$.

2.4 Invariant States

Under a natural condition on the parameters R, C, ν, ϑ, there exist fluid model solutions that are time invariant. Following Gromoll and Williams [8] (see Section 6), we call these invariant states for the fluid model.

Definition 2.3. *A vector of measures $\xi \in \mathbf{M}^{\mathbf{I}}$ is an invariant state for the fluid model if there is a fluid model solution ζ satisfying $\zeta(t) = \xi$ for all $t \ge 0$.*

To help describe invariant states, let

$$\mathcal{P} = \{z \in \mathbb{R}_+^{\mathbf{I}} : \phi_i(z) = \rho_i \text{ for all } i \in \mathcal{I}_+(z)\}. \tag{2.11}$$

Theorem 6.3 of Gromoll and Williams [8] gives necessary and sufficient conditions for the existence of invariant states for the fluid model and a representation for the invariant states. For convenience, we formulate these results as a proposition here and refer readers to [8] for the proof.

Proposition 2.1. *There exist invariant states for the fluid model if and only if*

$$R\rho \le C. \tag{2.12}$$

When (2.12) holds, the set of invariant states is given by

$$\mathcal{M} = \{\xi \in \mathbf{M}^{\mathbf{I}} : \xi_i = z_i \vartheta_i^e, \text{ for all } i \in \mathcal{I} \text{ and some } z \in \mathcal{P}\}. \tag{2.13}$$

Remark 2.6. We call $\mathcal{M}$ the invariant manifold for the fluid model.

2.5 Additional Notation for Fluid Model Solutions

Given a fluid model solution ζ, we let

$$\overline{M}_t^i(x) = \langle \mathbb{1}_{(x,\infty)}, \zeta_i(t) \rangle \quad \text{for all } t \ge 0,\ x \in \mathbb{R}_+, i \in \mathcal{I}.$$

Let (z, Λ) be auxiliary functions associated with ζ, as in Definition 2.1. For each $i \in \mathcal{I}$ and $0 \le s < t < \infty$, let

$$S_{s,t}^i = \int_s^t \frac{\Lambda_i(r)}{z_i(r)} \mathbb{1}_{(0,\infty)}\big(z_i(r)\big) dr. \tag{2.14}$$

Note that this may take the value $+\infty$. However, if $z_i(r) > 0$ for all $r \in [s,t]$, then $S_{s,t}^i < \infty$, since $\Lambda_i(\cdot)$ is bounded and $z_i(\cdot)$ is continuous (hence it is bounded away from zero on the interval $[s,t]$, being strictly positive there). Furthermore, in this case, $r \to S_{r,t}^i$ is continuously differentiable on $[s,t]$ because $\Lambda_i(\cdot) = \phi_i(z(\cdot))$ is continuous on $[s,t]$, since $z \to \phi_i(z)$ is continuous at points z where $z_i > 0$ (see Remark 2.1) and $r \to z_i(r)$ is continuous, and furthermore $r \to z_i(r)$ is continuous and bounded away from zero on $[s,t]$. We interpret $S_{s,t}^i$ as the cumulative amount of bandwidth per unit of fluid allocated to route i over the time interval $[s,t]$.

2.6 Some Properties of Fluid Model Solutions

The first two propositions in this subsection are the same as Corollary 2 and Lemma 6 in Section 5 of [6], respectively. For later use, we state the results here without proof.

Proposition 2.2. *Suppose that $\vartheta \in \mathbf{K}^{\mathbf{I}}$ and that ζ is a fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}$. Then $\zeta(t) \in \mathbf{K}^{\mathbf{I}}$ for all $t \geq 0$.*

Remark 2.7. The assumption on ϑ is in addition to the basic requirements that its components do not charge the origin and have finite first and second moments. The assumption on ϑ in Proposition 2.2 is automatically satisfied if our Assumption 2 (stated in Section 3.1) holds.

Proposition 2.3. *Suppose that ζ is a fluid model solution, $i \in \mathcal{I}$ and $0 \leq s < t < \infty$ such that $\zeta_i(r) \neq 0$ for all $r \in [s,t]$. Then*

$$\overline{M}^i_t(x) = \overline{M}^i_s(x + S^i_{s,t}) + \nu_i \int_s^t \overline{N}_i(x + S^i_{u,t})du \quad \textit{for all } x \in \mathbb{R}_+. \tag{2.15}$$

Remark 2.8. If ζ is a fluid model solution, $i \in \mathcal{I}$ and $0 \leq s_0 < t < \infty$ such that $\zeta_i(r) \neq 0$ for all $r \in (s_0, t]$ and $\zeta_i(s_0) = 0$, then (2.15) holds for $s \in (s_0, t]$ and letting $s \downarrow s_0$, since $\overline{M}^i_s(x+S^i_{s,t}) \leq \overline{M}^i_s(0) = z_i(s) \to z_i(s_0) = 0$ as $s \to s_0$, by taking the limit as $s \to s_0$ in (2.15), we obtain

$$\overline{M}^i_t(x) = \nu_i \int_{s_0}^t \overline{N}_i(x + S^i_{u,t})du \quad \text{for all } x \in \mathbb{R}_+. \tag{2.16}$$

3 Key Assumptions and Functions for Fluid Model Analysis

In this section, we first state additional assumptions on fluid model parameters for the critical case and on file size distributions needed for our analysis. Then we introduce functions H, K and $\underline{F}$, which are used in defining our Lyapunov function and establishing its properties. We describe some properties of $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$, the compositions of H and K, respectively, with a fluid model solution, ζ. In particular, we give the relationship between $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$, and some properties of $\underline{F}$. We characterize the optimizing solutions for the optimization problems used to define the bandwidth sharing policy and $\underline{F}$. The section concludes with further characterizations of the set of invariant states.

3.1 Key Assumptions

3.1.1 Critical Parameters

For our main results, we shall assume that the fluid model is critical, that is, the parameters (R, ρ, C) satisfy the following assumption.

Assumption 1. *We assume that*

$$\sum_{i \in \mathcal{I}} R_{ji}\rho_i \leq C_j \quad \text{for all } j \in \mathcal{J}, \tag{3.1}$$

and that $\mathcal{J}_* = \{j \in \mathcal{J} : \sum_{i\in\mathcal{I}} R_{ji}\rho_i = C_j\}$ is non-empty. Furthermore, without loss of generality, we assume that the first $\mathbf{J}_* = |\mathcal{J}_*|$ elements of $\mathcal{J}$ correspond to the set $\mathcal{J}_*$.

Assumption 1 requires that the average load on each resource is less than or equal to its capacity and that there exists at least one resource that is fully loaded.

Remark 3.1. The Lyapunov function defined later in this paper could also be applied when $\mathcal{J}_*$ is empty. Since the stability result for that strictly subcritical case has already been shown by Fu and Williams [6] with weaker assumptions, we focus only on the critical case here, where at least one resource is fully loaded.

3.1.2 File Size Distributions

The following assumption will be used in the proofs of Lemmas 3.1 and 3.2, which are used to prove Lemma 3.3. The latter gives the continuity in time of $\mathcal{H}^\zeta$, the composition of the function H (defined below) with a suitable fluid model solution ζ. This continuity property ultimately features in our proof of the absolute continuity of $\mathcal{H}^\zeta$ as a function of time and the convergence of fluid model solutions to the invariant manifold.

Assumption 2. *For each $i \in \mathcal{I}$, the file size distribution ϑ_i is assumed to be absolutely continuous with bounded hazard rate*[1]*. In particular, there is a finite constant C_ϑ such that $\overline{N}_i(x) \le C_\vartheta \overline{N}^e_i(x)$ for all $x \in [0,\infty)$, $i \in \mathcal{I}$.*

Remark 3.2. We already assumed in Section 2.1 that ϑ_i has finite first and second moments and Assumption 2 is in addition to this. For the definition and some examples related to hazard rate, see Appendix A.

Assumption 2 is used to prove the following lemma, which will help us to analyze the asymptotic behavior of fluid model solutions.

Lemma 3.1. *Suppose that Assumption 2 holds. Fix $T > 0$ and $\upsilon > 0$. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_\upsilon$, we have $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon^*_T}$ for all $t \in [0,T]$, where $\upsilon^*_T = \upsilon + C_\vartheta T \max_{i\in\mathcal{I}} \nu_i$.*

Proof. Let ζ be a fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_\upsilon$. For $t \in (0,T]$ and $i \in \mathcal{I}$, either $z_i(t) = 0$ or $z_i(t) \ne 0$. If $z_i(t) = 0$, then $\frac{\overline{M}^i_t(x)}{\overline{N}^e_i(x)} = 0$ for all $x \in [0,\infty)$. If $z_i(t) \ne 0$, let $t^i_0 = \sup\{s \in [0,t) : z_i(s) = 0\}$ where $\sup\emptyset = 0$. We consider the case where $t^i_0 > 0$ first. Then $\zeta_i(\cdot)$ is nonzero on $(t^i_0, t]$, $\zeta_i(t^i_0) = 0$ and by Remark 2.8 and Assumption 2, for all $x \in [0,\infty)$,

$$\frac{\overline{M}^i_t(x)}{\overline{N}^e_i(x)} \le \nu_i \frac{\overline{N}_i(x)}{\overline{N}^e_i(x)}(t - t^i_0) \le \nu_i C_\vartheta (t - t^i_0). \tag{3.2}$$

If $t^i_0 = 0$, then $\zeta(\cdot)$ is nonzero on $(0,t]$. In this case, by (2.15), for all $s \in (0,t)$, we have for all $x \in [0,\infty)$,

$$\overline{M}^i_t(x) \le \overline{M}^i_s(x) + \nu_i \overline{N}_i(x)(t-s).$$

On letting $s \downarrow 0$ and using the facts that $s \to \zeta(s)$ is continuous and $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_\upsilon$, together with Assumption 2, we obtain

$$\frac{\overline{M}^i_t(x)}{\overline{N}^e_i(x)} \le \upsilon + \nu_i C_\vartheta t, \text{ for all } x \in [0,\infty).$$

Combining the above with the fact that $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_\upsilon$, we obtain for any $t \in [0,T]$, $\frac{\overline{M}^i_t(x)}{\overline{N}^e_i(x)} \le \upsilon^*_{i,T}$ for all $x \in [0,\infty)$, where $\upsilon^*_{i,T} = \upsilon + \nu_i C_\vartheta T$. The desired result follows from this, Proposition 2.2 and the alternative representation of $\mathbf{K}^{\mathbf{I}}_{\upsilon^*_T}$ (see (2.8)). ☐

Remark 3.3. In Lemma 3.1, υ^*_T depends on T. Later, after more results have been developed, we shall prove in Lemma 8.2, with the addition of Assumption 1, that υ^*_T can be chosen not to depend on T.

3.2 Functions for Fluid Model Analysis

In this subsection, we shall define functions H and K on $\bigcup_{\upsilon>0} \mathbf{M}^{\mathbf{I}}_\upsilon$ and then apply them to fluid model solutions ζ with initial conditions in $\bigcup_{\upsilon>0} \mathbf{K}^{\mathbf{I}}_\upsilon$ to obtain functions $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$ of time. The larger domain for H and K is needed for the proof of Theorem 5.2.

[1] This condition implies the support of the distribution is unbounded.

3.2.1 The Functions H and $\mathcal{H}^\zeta$

Definition 3.1. *Given $\xi \in \cup_{\upsilon>0}\mathbf{M}^{\mathbf{I}}_\upsilon$, for each $i \in \mathcal{I}$, define*

$$H_i(\xi) = \frac{\kappa_i}{\rho_i^\alpha} \int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \right)^{\alpha+1} \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle dx, \tag{3.3}$$

and define

$$H(\xi) = \frac{1}{\alpha+1} \sum_{i\in\mathcal{I}} H_i(\xi). \tag{3.4}$$

Remark 3.4. For $\xi \in \bigcup_{\upsilon>0} \mathbf{M}^{\mathbf{I}}_\upsilon$, if $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle = 0$, then $\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle = 0$ and we interpret the integrand in (3.3) at x as being zero. Note that when Assumption 2 holds, $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle > 0$ for all $x \in [0,\infty)$, since the support of ϑ_i is unbounded in this case.

The function H will be used in defining our Lyapunov function. For $\xi \in \cup_{\upsilon>0}\mathbf{M}^{\mathbf{I}}_\upsilon$, there is $\upsilon > 0$ such that $\xi \in \mathbf{M}^{\mathbf{I}}_\upsilon$ and then $H_i(\xi) \le \frac{\kappa_i \upsilon^{\alpha+1}}{\rho_i^\alpha} \langle \chi, \vartheta_i^e \rangle < \infty$ for all $i \in \mathcal{I}$. It follows that $H_i(\xi)$, $i \in \mathcal{I}$, and $H(\xi)$ are finite. Furthermore, we have the following lemma.

Lemma 3.2. *The functions $H_i, i \in \mathcal{I}$, and H are continuous, non-negative, real-valued functions on $\mathbf{M}^{\mathbf{I}}_\upsilon$ for each $\upsilon > 0$.*

Proof. The non-negative, real-valued property follows from observation and the last paragraph before this lemma. For the continuity, fix $\upsilon > 0$. Suppose that $\{\xi_n\}_{n=1}^\infty$ is a sequence in $\mathbf{M}^{\mathbf{I}}_\upsilon$ converging (weakly) to $\xi \in \mathbf{M}^{\mathbf{I}}_\upsilon$. Then as $n \to \infty$, $\langle \mathbb{1}_{(x,\infty)}, \xi_n \rangle \to \langle \mathbb{1}_{(x,\infty)}, \xi \rangle$ for almost every $x \in [0,\infty)$. Since $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathbf{M}^{\mathbf{I}}_\upsilon$, the sequence of integrands in the definition of $H_i(\xi_n)$ is dominated by $\upsilon^{\alpha+1}\overline{N}_i^e(\cdot)$, which is integrable because $\langle \chi, \vartheta_i^e \rangle < \infty$. Thus, by the dominated convergence theorem, $H_i(\xi_n) \to H_i(\xi)$ as $n \to \infty$ for each $i \in \mathcal{I}$. It follows that $H_i, i \in \mathcal{I}$ and H are continuous on $\mathbf{M}^{\mathbf{I}}_\upsilon$. □

Remark 3.5. The form of H is largely inspired by two prior works: Mulvany et al. [19] and Paganini et al. [20]. In [19], building on work of Puha and Williams [23], Mulvany et al. considered a relative entropy functional for comparing the probability measure on $\mathbb{R}_+$ with density proportional to $p_i(x) = \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle$ to the probability measure on $\mathbb{R}_+$ with density proportional to $q_i(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$. When normalized to be probability densities, p_i and q_i are the densities of excess lifetime distributions associated with ξ_i and ϑ_i^e, respectively. The relative entropy employed by Mulvany et al. [19] uses $u \to u\ln(u)$ in place of the function $f(u) = u^{\alpha+1}$ that we have used in the integral in (3.3). The form of $H_i(\xi)$ used here is proportional to the so-called f-divergence [4] for the two finite measures on $\mathbb{R}_+$ that have densities p_i and q_i. Further inspiration for our use of f in place of $u \to u\ln(u)$ comes from Paganini et al. [20]; see also Fu and Williams [6] for the inclusion of the weights κ_i. In those works, for the strictly subcritical case, f was applied directly to the function p_i (no quotient) and integrated with a reference density θ_i that involved ϑ_i^e, to give the i^{th} Lyapunov function component. In fact, if one formally takes the limit in the Lyapunov function in [20, 6] as critical loading is approached on all resources, one obtains the H_i and H in (3.3) and (3.4) for the case where equality holds in (2.12) (all resources are fully loaded).

Definition 3.2. *Suppose that Assumption 2 holds. Given a fluid model solution ζ with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_\upsilon$, for each $t \ge 0$ and $i \in \mathcal{I}$, define*

$$\mathcal{H}_i^\zeta(t) = H_i\big(\zeta(t)\big) = \frac{\kappa_i}{\rho_i^\alpha} \int_0^\infty \left(\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \right)^{\alpha+1} \overline{N}_i^e(x) dx \quad \text{for all } i \in \mathcal{I}, \tag{3.5}$$

and let

$$\mathcal{H}^\zeta(t) = H\big(\zeta(t)\big) = \frac{1}{\alpha+1} \sum_{i\in\mathcal{I}} \mathcal{H}_i^\zeta(t). \tag{3.6}$$

Lemma 3.3. *Suppose that Assumption 2 holds. Let ζ be a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_\upsilon$. Then for each $i \in \mathcal{I}$, $\mathcal{H}_i^\zeta : [0,\infty) \to [0,\infty)$ is well defined and continuous on $[0,\infty)$.*

Proof. This follows immediately on combining Lemmas 3.1, 3.2 and the fact that $t \to \zeta(t)$ is continuous. □

3.2.2 The Functions K and $\mathcal{K}^\zeta$

In this section, we introduce the functions K and $\mathcal{K}^\zeta$. The latter arises in taking the derivative of the function $\mathcal{H}^\zeta(\cdot)$.

Definition 3.3. *Given $\xi \in \cup_{\upsilon>0}\mathbf{M}_\upsilon^{\mathbf{I}}$, for each $i \in \mathcal{I}$, define*

$$K_i(\xi) = \kappa_i\rho_i^{-\alpha}\bigg(-\phi_i(\langle \mathbb{1},\xi\rangle)(\langle \mathbb{1},\xi_i\rangle)^\alpha + \int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle}\right)^\alpha \langle \mathbb{1}_{(x,\infty)},\vartheta_i\rangle\Big(-\frac{\alpha\phi_i(\langle \mathbb{1},\xi\rangle)}{\langle \mathbb{1},\xi_i\rangle}\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle\langle\chi,\vartheta_i\rangle} + \nu_i(\alpha+1)\Big)\mathbb{1}_{(0,\infty)}(\langle \mathbb{1},\xi_i\rangle)dx\bigg). \tag{3.7}$$

Then with $z = \langle \mathbb{1},\xi\rangle$, define

$$K(\xi) = \frac{1}{\alpha+1}\sum_{i\in\mathcal{I}_+(z)} K_i(\xi). \tag{3.8}$$

Remark 3.6. For $\xi \in \bigcup_{\upsilon>0}\mathbf{M}_\upsilon^{\mathbf{I}}$, if $x \in \mathbb{R}_+$ such that $\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle = 0$, then $\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle = 0$ and we interpret the integrand in the integral in (3.7) as being zero at x. In (3.7), if $\xi_i = 0$, we interpret the right member of the equality to be zero and so $K_i(\xi) = 0$ in this case. If $\xi_i \neq 0$, there is $\upsilon > 0$ such that $\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle \le \upsilon\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle$ for all $x \in [0,\infty)$. Then noticing $\int_0^\infty \langle \mathbb{1}_{(x,\infty)},\vartheta_i\rangle dx = \langle\chi,\vartheta_i\rangle < \infty$, we have $|K_i(\xi)| < \infty$. Note that (3.8) can also be written as $K(\xi) = \sum_{i\in\mathcal{I}} K_i(\xi)/(\alpha+1)$.

The following property of the K_i and K will be used in proving our main results.

Lemma 3.4. *Fix $\upsilon > 0$. The functions K_i, $i \in \mathcal{I}$, and K are real-valued, upper semicontinuous functions on $\mathbf{M}_\upsilon^{\mathbf{I}}$. Furthermore, if $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$ and $i \in \mathcal{I}$ such that $z_i = \langle \mathbb{1},\xi_i\rangle \neq 0$, then K_i is continuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ.*

Proof. The real-valuedness of $K_i, i \in \mathcal{I}$, and K follows from Remark 3.6. For $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$, let

$$\mathrm{k}_i^{(1)}(\xi) = -\kappa_i\rho_i^{-\alpha}\phi_i(\langle \mathbb{1},\xi\rangle)(\langle \mathbb{1},\xi_i\rangle)^\alpha,$$

$$\mathrm{k}_i^{(2)}(\xi) = -\kappa_i\rho_i^{-\alpha}\int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle}\right)^\alpha \langle \mathbb{1}_{(x,\infty)},\vartheta_i\rangle\left(\frac{\alpha\phi_i(\langle \mathbb{1},\xi\rangle)}{\langle \mathbb{1},\xi_i\rangle}\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle\langle\chi,\vartheta_i\rangle}\right)\mathbb{1}_{(0,\infty)}(\langle \mathbb{1},\xi_i\rangle)dx,$$

$$\mathrm{k}_i^{(3)}(\xi) = \kappa_i\rho_i^{-\alpha}\int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle}\right)^\alpha \langle \mathbb{1}_{(x,\infty)},\vartheta_i\rangle\nu_i(\alpha+1)\mathbb{1}_{(0,\infty)}(\langle \mathbb{1},\xi_i\rangle)dx.$$

Fix $\xi \in \mathbf{M}_\upsilon^{\mathbf{I}}$ and let $z = \langle \mathbb{1},\xi\rangle$. We first show that, for each $i \in \mathcal{I}_+(z)$, K_i is continuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ. Fix $i \in \mathcal{I}_+(z)$. Suppose $\{\xi^n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbf{M}_\upsilon^{\mathbf{I}}$ that converges to ξ (weakly). We want to show that $\lim_{n\to\infty} K_i(\xi^n) = K_i(\xi)$. For $\mathrm{k}_i^{(1)}$, by the continuity of $\phi_i(\cdot)$ at z when $z_i \neq 0$, and the fact that ξ^n converges to ξ implying $\langle \mathbb{1},\xi^n\rangle \to \langle \mathbb{1},\xi\rangle$, we have $\lim_{n\to\infty} \mathrm{k}_i^{(1)}(\xi^n) = \mathrm{k}_i^{(1)}(\xi)$. For $\mathrm{k}_i^{(2)}$, we have $\langle \mathbb{1}_{(x,\infty)},\xi_i^n\rangle \to \langle \mathbb{1}_{(x,\infty)},\xi_i\rangle$ as $n \to \infty$ for almost every $x \in [0,\infty)$, $\langle \mathbb{1},\xi_i^n\rangle \to z_i \neq 0$ and $\phi_i(\langle \mathbb{1},\xi^n\rangle) \to \phi_i(\langle \mathbb{1},\xi\rangle)$ as $n \to \infty$ (by the continuity of ϕ_i at z such that $z_i \neq 0$), $\frac{\langle \mathbb{1}_{(x,\infty)},\xi_i^n\rangle}{\langle \mathbb{1}_{(x,\infty)},\vartheta_i^e\rangle} \le \upsilon$ for all $n \in \mathbb{N}$ and x such that $\langle \mathbb{1}_{(x,\infty)},\vartheta_i\rangle > 0$, and $\int_0^\infty \langle \mathbb{1}_{(x,\infty)},\vartheta_i\rangle dx = \langle\chi,\vartheta_i\rangle < \infty$, and so using the fact that $\phi_i(\langle \mathbb{1},\xi^n\rangle) \le \max_{j\in\mathcal{J}} C_j$ for all $n \in \mathbb{N}$, we can apply the dominated convergence theorem to conclude that $\lim_{n\to\infty} \mathrm{k}_i^{(2)}(\xi^n) = \mathrm{k}_i^{(2)}(\xi)$. For $\mathrm{k}_i^{(3)}$, we can also apply the dominated convergence theorem to conclude that $\lim_{n\to\infty} \mathrm{k}_i^{(3)}(\xi^n) = \mathrm{k}_i^{(3)}(\xi)$. It follows that $K_i = \mathrm{k}_i^{(1)} + \mathrm{k}_i^{(2)} + \mathrm{k}_i^{(3)}$, is continuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ for $i \in \mathcal{I}_+(z)$. This proves the last statement of the lemma.

For $i \in \mathcal{I}\backslash\mathcal{I}_+(z)$, we will show that K_i is upper semicontinuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ, where $\xi_i = 0$. For this it suffices to show for $\{\xi^n\}_{n\in\mathbb{N}}$, a sequence in $\mathbf{M}_\upsilon^{\mathbf{I}}$ that converges to ξ (weakly), we have $\limsup_{n\to\infty} K_i(\xi^n) \le K_i(\xi)$. Notice that $\mathrm{k}_i^{(1)}(\xi^n) \le 0$ and $\mathrm{k}_i^{(2)}(\xi^n) \le 0$, while $\mathrm{k}_i^{(1)}(\xi) = 0$ and $\mathrm{k}_i^{(2)}(\xi) = 0$. It follows that $\limsup_{n\to\infty}(\mathrm{k}_i^{(1)}(\xi^n) + \mathrm{k}_i^{(2)}(\xi^n)) \le \mathrm{k}_i^{(1)}(\xi) + \mathrm{k}_i^{(2)}(\xi)$. For $\mathrm{k}_i^{(3)}(\xi^n)$, the integrand is dominated by the integrable function $x \to \upsilon^\alpha\nu_i(\alpha+1)\langle \mathbb{1}_{(x,\infty)},\vartheta_i\rangle$ and tends to zero as $n \to \infty$, since $\langle \mathbb{1}_{(x,\infty)},\xi_i^n\rangle \le z_i^n \to z_i = 0$ as $n \to \infty$. It follows by the dominated convergence theorem that $\mathrm{k}_i^{(3)}(\xi^n) \to 0 = \mathrm{k}_i^{(3)}(\xi)$ as $n \to \infty$. Combining, we see that K_i is upper semicontinuous on $\mathbf{M}_\upsilon^{\mathbf{I}}$ at ξ, for $i \notin \mathcal{I}_+(z)$.

Since $\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon}$ was arbitrary and any continuous function is upper semicontinuous, it follows that K_i is upper semicontinuous on $\mathbf{M}^{\mathbf{I}}_{\upsilon}$ for each $i \in \mathcal{I}$. Furthermore, $K = \sum_{i\in\mathcal{I}} K_i/(\alpha+1)$ is upper semicontinuous on $\mathbf{M}^{\mathbf{I}}_{\upsilon}$, being a linear combination, with positive coefficients, of such functions. $\square$

The following is a key lemma, proved in Section 6.1. For the statement of this, let δ_0 denote the probability measure on $\mathbb{R}_+$ that has unit mass at the origin and define

$$\mathcal{M}^* = \{\xi \in \mathbf{M}^{\mathbf{I}} : \text{ for each } i \in \mathcal{I},\ \xi_i = a_i\delta_0 + b_i\vartheta^e_i, \text{ where } a \in \mathbb{R}^{\mathbf{I}}_+ \text{ and } b \in \mathcal{P}\}. \tag{3.9}$$

Lemma 3.5. *Given $\xi \in \bigcup_{\upsilon>0} \mathbf{M}^{\mathbf{I}}_{\upsilon}$, with $z = \langle \mathbb{1}, \xi\rangle$ and $z_i = \langle \mathbb{1}, \xi_i\rangle$ for each $i \in \mathcal{I}$, we have*

$$K_i(\xi) \le \kappa_i z_i^\alpha \left(\frac{-\phi_i(z)}{\rho_i^\alpha} + \frac{\rho_i}{\big(\phi_i(z)\big)^\alpha}\right)\mathbb{1}_{(0,\infty)}(z_i). \tag{3.10}$$

Moreover, if Assumption 1 is satisfied, we have

$$K(\xi) \le \sum_{i\in\mathcal{I}_+(z)} \kappa_i \left(\frac{z_i}{\phi_i(z)}\right)^\alpha \big(\rho_i - \phi_i(z)\big) \le 0, \tag{3.11}$$

where equality holds everywhere in (3.11) if and only if $\xi \in \mathcal{M}^*$.

Definition 3.4. *Suppose that Assumption 2 holds. Given a fluid model solution ζ with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_{\upsilon}$, for each $t \ge 0$ and $i \in \mathcal{I}$, define*

$$\begin{aligned}\mathcal{K}^\zeta_i(t) = K_i\big(\zeta(t)\big) = \frac{\kappa_i}{\rho_i^\alpha}\Bigg(&-\Lambda_i(t)\big(z_i(t)\big)^\alpha \\ &+ \int_0^\infty \left(\frac{\overline{M}^i_t(x)}{\overline{N}^e_i(x)}\right)^\alpha \overline{N}_i(x)\left(-\frac{\alpha\Lambda_i(t)}{z_i(t)}\frac{\overline{M}^i_t(x)}{\overline{N}^e_i(x)\langle\chi,\vartheta_i\rangle} + \nu_i(\alpha+1)\right)\mathbb{1}_{(0,\infty)}\big(z_i(t)\big)dx\Bigg)\end{aligned} \tag{3.12}$$

and

$$\mathcal{K}^\zeta(t) = \frac{1}{\alpha+1}\sum_{i\in\mathcal{I}_+(z(t))} \mathcal{K}^\zeta_i(t) \text{ for all } t \ge 0. \tag{3.13}$$

Lemma 3.6. *Suppose that Assumption 2 holds. Let ζ be a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_{\upsilon}$. Then $\mathcal{K}^\zeta_i, i \in \mathcal{I}$, and $\mathcal{K}^\zeta$ are real-valued, upper semicontinuous functions on $[0,\infty)$. Furthermore, for each $i \in \mathcal{I}$, $\mathcal{K}^\zeta_i$ is continuous on $\{t \ge 0 : z_i(t) > 0\}$.*

Proof. This follows immediately on combining Lemma 3.1 with Lemma 3.4 and the continuity of $\zeta(\cdot)$ on $[0,\infty)$. $\square$

3.2.3 Relationship between $\mathcal{H}^\zeta$ and $\mathcal{K}^\zeta$

Theorem 3.1. *Suppose that Assumptions 1 and 2 hold. Further suppose that ζ is a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_{\upsilon}$. For each $i \in \mathcal{I}$, $\mathcal{K}^\zeta_i(\cdot)$ is integrable over $[0,t]$ for each $t \ge 0$ and the function $\mathcal{H}^\zeta_i(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$, with density $\mathcal{K}^\zeta_i(\cdot)$, and so*

$$\mathcal{H}^\zeta_i(t) - \mathcal{H}^\zeta_i(0) = \int_0^t \mathcal{K}^\zeta_i(s)ds \quad \text{for each } t \ge 0. \tag{3.14}$$

Consequently, $\mathcal{H}^\zeta(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$ and $\mathcal{K}^\zeta(\cdot)$ is a density for $\mathcal{H}^\zeta(\cdot)$. Furthermore, for each $t \ge 0$,

$$\mathcal{K}^\zeta(t) \le \sum_{i\in\mathcal{I}_+(z(t))} \kappa_i \left(\frac{z_i(t)}{\Lambda_i(t)}\right)^\alpha \big(\rho_i - \Lambda_i(t)\big) \le 0, \tag{3.15}$$

where equality holds everywhere in (3.15) if and only if $\zeta(t) \in \mathcal{M}$. Hence $\mathcal{H}^\zeta(\cdot)$ is non-increasing on $[0,\infty)$, and is strictly decreasing at times $t \in [0,\infty)$ where $\zeta(t) \notin \mathcal{M}$.

The proof of Theorem 3.1 is given in Section 6.

3.2.4 The Function $\underline{F}$

One characterization of the invariant states for the fluid model that we will give uses the following optimization problem. This optimization problem is similar to one used by Kelly and Williams [14], who studied properties of the fluid model when ϑ_i is exponentially distributed for each $i \in \mathcal{I}$. The main difference in the form from [14] is that in two places (one in the function F and one in the constraint of the optimization problem (3.16)), $\frac{1}{\mu_i}$ from [14] is replaced by $\langle \chi, \vartheta_i^e \rangle$ for $i \in \mathcal{I}$. We now describe the optimization problem.

For $z \in \mathbb{R}_+^{\mathbf{I}}$, let

$$F(z) = \frac{1}{\alpha+1} \sum_{i \in \mathcal{I}} \frac{\kappa_i \langle \chi, \vartheta_i^e \rangle}{\rho_i^\alpha} z_i^{\alpha+1}.$$

For $\widetilde{w} \in \mathbb{R}_+^{\mathbf{J}_*}$, consider the optimization problem

$$\text{minimize } F(z) \text{ subject to } \sum_{i \in \mathcal{I}} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle \geq \widetilde{w}_j \text{ for all } j \in \mathcal{J}_* \text{ and } z \in \mathbb{R}_+^{\mathbf{I}}. \tag{3.16}$$

In Section 3.4, we give several different characterizations of the set $\mathcal{P}$, which features in the characterization (2.13) of invariant states for the fluid model. One of these uses the optimization problem (3.16). For $\widetilde{w} \in \mathbb{R}_+^{\mathbf{J}_*}$, let $\underline{F}(\widetilde{w})$ be the optimal value attained in the optimization problem (3.16) and let $\Delta(\widetilde{w})$ be the optimizing value of z. These exist and are unique. The following proposition gives properties of $\underline{F}$. Its proof is the same as that of Lemma 6.3 of [14] with $diag(\langle \chi, \vartheta_e \rangle)$ in place of $M^{-1} = diag(\mu_i^{-1} : i \in \mathcal{I})$, and we refer the reader to [14] for the details. We note that this proof uses the fact that R has full row rank.

Proposition 3.1. *The functions $\underline{F} : \mathbb{R}_+^{\mathbf{J}_*} \to \mathbb{R}_+$ and $\Delta : \mathbb{R}_+^{\mathbf{J}_*} \to \mathbb{R}_+^{\mathbf{I}}$ are continuous. In addition, $\underline{F}$ is a non-decreasing function, i.e., for $\widetilde{w}, \widetilde{w}^\dagger \in \mathbb{R}_+^{\mathbf{J}_*}$, if $\widetilde{w}_j \leq \widetilde{w}_j^\dagger$ for each $j \in \mathcal{J}_*$, then $\underline{F}(\widetilde{w}) \leq \underline{F}(\widetilde{w}^\dagger)$.*

The non-decreasing property of $\underline{F}$ will be a key property for proving that our Lyapunov function, when applied to a fluid model solution, yields a non-increasing function of time.

3.3 Characterization of Solutions for the Optimization Problems (2.3) and (3.16)

The first optimization problem considered here is (2.3) and the second is (3.16). We characterize the optimal solutions for both problems below, so as to give an alternative characterization of the invariant states. The idea of using these two optimization problems to characterize invariant states was employed by Kelly and Williams [14] when the file sizes are exponentially distributed. Proposition 3.2, which characterizes the optimal solution for (2.3), is equivalent to Lemma A.4 in [14]. Proposition 3.3, which characterizes the optimal solution for (3.16), is similar to Lemma 6.4 in [14]. Both propositions are proved using Lagrange multipliers. For the proof of Proposition 3.3, in the proof of Lemma 6.4 in [14], substitute $\langle \chi, \vartheta_e^i \rangle$ for μ_i^{-1} in the constraints and in one place in F. The proof uses the fact that R has full row rank. We refer readers to [14] for details of the proofs of these two propositions.

Proposition 3.2. *Fix $z \in \mathbb{R}_+^{\mathbf{I}} \backslash \{0\}$, where 0 is the origin of $\mathbb{R}_+^{\mathbf{I}}$. A vector $\psi = (\psi_i : i \in \mathcal{I}) \in \mathcal{O}(z)$ is the unique optimal solution of (2.3), i.e. $\psi = \phi(z)$, if and only if there is $p \in \mathbb{R}_+^{\mathbf{J}}$ such that*

$$p_j \big(C_j - \sum_{i \in \mathcal{I}_+(z)} R_{ji} \psi_i \big) = 0 \quad \textit{for all } j \in \mathcal{J}, \tag{3.17}$$

$$\sum_{j \in \mathcal{J}} p_j R_{ji} > 0 \quad \textit{for all } i \in \mathcal{I}_+(z), \tag{3.18}$$

$$\psi_i = z_i \Big(\frac{\kappa_i}{\sum_{j \in \mathcal{J}} p_j R_{ji}} \Big)^{1/\alpha} \quad \textit{for all } i \in \mathcal{I}_+(z) \quad \textit{and} \tag{3.19}$$

$$\sum_{i \in \mathcal{I}_+(z)} R_{ji} \psi_i \leq C_j \quad \textit{for all } j \in \mathcal{J}. \tag{3.20}$$

Proposition 3.3. *Suppose Assumption 1 holds. For each $\widetilde{w} \in \mathbb{R}_+^{\mathbf{J}_*}$, a vector $z \in \mathbb{R}_+^{\mathbf{I}}$ is the unique optimal solution of (3.16), i.e. $z = \Delta(\widetilde{w})$, if and only if there is $p \in \mathbb{R}_+^{\mathbf{J}_*}$ such that for each $i \in \mathcal{I}$,*

$$z_i = \rho_i \left(\frac{\sum_{j \in \mathcal{J}_*} p_j R_{ji}}{\kappa_i} \right)^{1/\alpha}, \tag{3.21}$$

and for each $j \in \mathcal{J}_*$,

$$p_j \left(\sum_{i \in \mathcal{I}} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle - \widetilde{w}_j \right) = 0 \quad \textit{and} \quad \sum_{i \in \mathcal{I}} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle \geq \widetilde{w}_j.$$

3.4 Further Characterizations of Invariant States

Under Assumption 1, recall the set of invariant states $\mathcal{M}$ is given by (2.13) and $\mathcal{P}$ is defined in (2.11). Here we characterize the set $\mathcal{P}$ in two further ways, similar to Lemma 6.4 of [8], whose proof relies on those of Theorems 5.1 and 5.3 of [14].

Lemma 3.7. *Suppose Assumption 1 holds. The following three conditions are equivalent:*

(i) $z \in \mathcal{P}$,

(ii) for some $p \in \mathbb{R}_+^{\mathbf{J}_*}$, $z_i = \rho_i \big(\frac{1}{\kappa_i} \sum_{j \in \mathcal{J}_*} p_j R_{ji} \big)^{1/\alpha}$ *for all* $i \in \mathcal{I}$,

(iii) $z = \Delta \left(\check{w}(z) \right)$, *where* $\check{w}_j(z) = \sum_{i \in \mathcal{I}} R_{ji} z_i \langle \chi, \vartheta_i^e \rangle$ *for all* $j \in \mathcal{J}_*$.

Proof. The proof is very similar to that of Theorems 5.1 and 5.3 of [14], with $\langle \chi, \vartheta_i^e \rangle$ replacing μ_i^{-1} in two places for each $i \in \mathcal{I}$. For *(i)* $\Leftrightarrow$ *(ii)*, one uses Proposition 3.2; and for *(ii)* $\Leftrightarrow$ *(iii)*, one uses Proposition 3.3 in place of Lemma 6.4 of [14]. ☐

Remark 3.7. The above characterization of $\mathcal{P}$ is slightly different from what is given by Gromoll and Williams [8]. The latter uses $w(z)$ and μ_i^{-1} in the constraints rather than $\check{w}(z)$ and $\langle \chi, \vartheta_i^e \rangle$. Both characterizations are correct and although the difference is subtle, we find that our form is more useful for our proofs.

4 Lyapunov Function G and $\mathcal{G}^\zeta$

In this section, we define the Lyapunov function G on $\bigcup_{\upsilon > 0} \mathbf{M}_\upsilon^{\mathbf{I}}$ and the function $\mathcal{G}^\zeta$ for any fluid model solution satisfying $\zeta(0) \in \bigcup_{\upsilon > 0} \mathbf{K}_\upsilon^{\mathbf{I}}$.

Definition 4.1. *Given* $\xi \in \bigcup_{\upsilon > 0} \mathbf{M}_\upsilon^{\mathbf{I}}$, *define*

$$G(\xi) = H(\xi) - \underline{F}\big(\widetilde{w}(\xi)\big) \tag{4.1}$$

where $\widetilde{w}_j(\xi) = \sum_{i \in \mathcal{I}} R_{ji} \langle \chi, \xi_i \rangle$ *for each* $j \in \mathcal{J}_*$, *and* $\underline{F}\left(\widetilde{w}\left(\xi\right)\right)$ *is the optimal value for the optimization problem (3.16) with* $\widetilde{w} = \widetilde{w}(\xi)$.

The following lemma is proved in Section 7.3.

Lemma 4.1. *For each* $\upsilon > 0$,

(i) $G : \mathbf{M}_\upsilon^{\mathbf{I}} \to [0, \infty)$ *is continuous.*

Moreover, if Assumption 1 holds, then for any $\xi \in \cup_{\upsilon > 0} \mathbf{M}_\upsilon^{\mathbf{I}}$,

(ii) $G(\xi) = 0$ *if and only if* $\xi \in \mathcal{M}^*$, *where* $\mathcal{M}^*$ *is given by (3.9).*

Definition 4.2. *Suppose that Assumption 2 holds. Given a fluid model solution ζ with* $\zeta(0) \in \bigcup_{\upsilon > 0} \mathbf{K}_\upsilon^{\mathbf{I}}$, *define*

$$\mathcal{G}^\zeta(t) = G(\zeta(t)) = \mathcal{H}^\zeta(t) - \underline{F}(\widetilde{w}(\zeta(t))) \text{ for all } t \geq 0. \tag{4.2}$$

Remark 4.1. By Lemmas 3.1 and 4.1, $\mathcal{G}^\zeta(\cdot)$ is well defined and continuous on $[0, \infty)$.

5 Main Results

The proofs of the next three theorems are given in Section 8.

Theorem 5.1. *Suppose that Assumptions 1 and 2 hold. Further suppose that ζ is a fluid model solution with $\zeta(0) \in \cup_{\upsilon>0}\mathbf{K}^{\mathbf{I}}_{\upsilon}$. Then*

(i) $\mathcal{G}^{\zeta} : [0,\infty) \to [0,\infty)$ *is continuous,*

(ii) for any $t \geq 0$, $\mathcal{G}^{\zeta}(t) = 0$ if and only if $\zeta(t) \in \mathcal{M}$, and

(iii) $\mathcal{G}^{\zeta}$ *is a non-increasing function on $[0,\infty)$ and at times $t \in [0,\infty)$ where $\zeta(t) \notin \mathcal{M}$, $\mathcal{G}^{\zeta}$ is strictly decreasing.*

Theorem 5.2. *Suppose that Assumptions 1 and 2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, $\mathcal{G}^{\zeta}(t)$ decreases monotonically to zero as $t \to \infty$. Furthermore, this convergence is uniform, i.e.,*

$$\lim_{t\to\infty} \sup\{\mathcal{G}^{\zeta}(t) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}\} = 0. \tag{5.1}$$

Theorem 5.3. *Suppose that Assumptions 1 and 2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, $\zeta(t)$ converges towards $\mathcal{M}$ as $t \to \infty$, uniformly for all initial measures in $\mathbf{K}^{\mathbf{I}}_{\upsilon}$, i.e.,*

$$\lim_{t\to\infty} \sup\{\mathbf{d_I}(\zeta(t),\mathcal{M}) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}\} = 0. \tag{5.2}$$

Furthermore, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\sup_{t\geq 0}\{\mathbf{d_I}(\zeta(t),\mathcal{M}) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon} \text{ and } \mathbf{d_I}(\zeta(0),\mathcal{M}) < \delta\} \leq \epsilon. \tag{5.3}$$

6 Proofs of Lemma 3.5 and Theorem 3.1

6.1 Proof of Lemma 3.5

The following three propositions are needed for our proof of Lemma 3.5. They are nearly the same as Lemmas 1, 2 and 3 in Section 5 of Fu and Williams [6] (with $\alpha_i = \alpha$ for all $i \in \mathcal{I}$ in [6]), which are similar to Lemma 1, a result in Section III.C, and Lemma 5, respectively, of Paganini et al. [20]. Beyond what is covered by these prior works, for Proposition 6.3, we add a condition for equality in the inequality. We indicate the reasoning for that here and leave the reader to consult [6] for the proofs of Propositions 6.1, 6.2 and the rest of Proposition 6.3.

Proposition 6.1. *Fix $z \in \mathbb{R}^{\mathbf{I}}_{+}$. Recall that $\phi(z)$ is the unique maximizer for the optimization problem (2.3). Let $\psi \in \mathbb{R}^{\mathbf{I}}_{+}$ such that $\psi_i > 0$ for all $i \in \mathcal{I}_+(z)$ and $\sum_{i\in\mathcal{I}} R_{ji}\psi_i \leq C_j$ for all $j \in \mathcal{J}$. Then*

$$\sum_{i\in\mathcal{I}_+(z)} \kappa_i U'\left(\frac{\phi_i(z)}{z_i}\right)\big(\psi_i - \phi_i(z)\big) \leq 0, \tag{6.1}$$

where U' is the derivative of U on $(0,\infty)$.

Proposition 6.2. *Let $g(s) = s^a\big((a+1)q - bs\big)$ for $s \geq 0$ where a, b, q are fixed strictly positive real numbers. Then g has a unique maximum of $\left(\frac{aq}{b}\right)^a q$ at $s = \frac{aq}{b}$.*

Proposition 6.3. *For any strictly positive real numbers, a, b, q, we have*

$$-\frac{b}{q^a} + \frac{q}{b^a} \leq (a+1)\frac{q-b}{b^a}, \tag{6.2}$$

where equality holds if and only if $q = b$.

Proof of when equality holds in (6.2). The inequality (6.2) comes from the fact that the tangent line to the graph of $y = x^{a+1}$ at $x = q$ is a lower support line for the graph. It follows from the strict convexity of $x \to x^{a+1}$ that this support line touches the graph only at $x = q$ and hence the inequality in (6.2) is strict for all $b \neq q$. □

Proof of Lemma 3.5. We first prove (3.10). Since both sides of the inequality are zero when $z_i = 0$, it suffices to consider the case where $z_i > 0$. In this case, we have

$$\begin{aligned}\rho_i^\alpha K_i(\xi) &= -\kappa_i \phi_i(z) z_i^\alpha \\ &\quad + \kappa_i \int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \right)^\alpha \left((\alpha+1)\rho_i - \frac{\alpha \phi_i(z)}{z_i} \frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} \right) \mu_i \overline{N}_i(x) dx \\ &\leq -\kappa_i \phi_i(z) z_i^\alpha + \kappa_i \int_0^\infty \left(\frac{\rho_i z_i}{\phi_i(z)} \right)^\alpha \rho_i \mu_i \overline{N}_i(x) dx \qquad (6.3) \\ &= \kappa_i z_i^\alpha \left(-\phi_i(z) + \frac{\rho_i^{\alpha+1}}{(\phi_i(z))^\alpha} \right), \qquad (6.4)\end{aligned}$$

where we used Proposition 6.2 with $a = \alpha$, $q = \rho_i$, $b = \frac{\alpha \phi_i(z)}{z_i}$ for $z_i > 0$ to obtain the inequality, and the fact that $\int_0^\infty \mu_i \overline{N}_i(x) dx = 1$ for the last equality. We note here that the inequality in (6.3) is strict unless $\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle = \frac{\rho_i z_i}{\phi_i(z)} \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$ for all $x \in \mathbb{R}_+$; this follows for $x \in \mathbb{R}_+$ where $\overline{N}_i(x) > 0$ by the uniqueness of the maximum in Proposition 6.2, and the relation automatically holds for x where $\overline{N}_i(x) = 0$, since $\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle = 0$ for such x and $\xi \in \mathbf{M}_v^{\mathbf{I}}$ for some $v > 0$. Thus,

$$\begin{aligned}K_i(\xi) &\leq \kappa_i z_i^\alpha \left(\frac{-\phi_i(z)}{\rho_i^\alpha} + \frac{\rho_i}{(\phi_i(z))^\alpha} \right) \qquad (6.5) \\ &\leq \kappa_i z_i^\alpha (\alpha+1) \frac{\rho_i - \phi_i(z)}{(\phi_i(z))^\alpha}, \qquad (6.6)\end{aligned}$$

where the last step follows by Proposition 6.3 with $a = \alpha, b = \phi_i(z)$ and $q = \rho_i$. We note here that by Proposition 6.3, the last inequality is strict unless $\phi_i(z) = \rho_i$. The inequality (6.5) yields (3.10).

Assuming that Assumption 1 holds, we shall now use inequality (6.6) to prove (3.11), and we shall use the conditions for equality in (6.3) and (6.6) to determine conditions for equality in (3.11). For $i \in \mathcal{I}_+(z)$, $U'\left(\frac{\phi_i(z)}{z_i}\right) = \left(\frac{z_i}{\phi_i(z)}\right)^\alpha$. Furthermore, ρ has positive components and satisfies $\sum_{i \in \mathcal{I}} R_{ji} \rho_i \leq C_j$ for all $j \in \mathcal{J}$, by Assumption 1. Then, by (6.6) and replacing $z, \psi, \phi(z)$ by $z, \rho, \phi(z)$, respectively, in Proposition 6.1, we obtain

$$K(\xi) = \frac{1}{\alpha+1} \sum_{i \in \mathcal{I}_+(z)} K_i(\xi) \leq \sum_{i \in \mathcal{I}_+(z)} \kappa_i \left(\frac{z_i}{\phi_i(z)} \right)^\alpha \big(\rho_i - \phi_i(z) \big) \leq 0. \qquad (6.7)$$

Hence (3.11) holds. By the conditions for equality in (6.3) and (6.6), the first inequality in (6.7) is an equality if and only if $\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle = \frac{\rho_i z_i}{\phi_i(z)} \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle$ for all $x \in \mathbb{R}_+$ and $\phi_i(z) = \rho_i$, for all $i \in \mathcal{I}_+(z)$. Noting that $z_i = 0$ for all $i \notin \mathcal{I}_+(z)$, it then follows that $K(\xi) = 0$ if and only if for all $i \in \mathcal{I}$, $\langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle = z_i \overline{N}_i^e(x)$ for all $x \in \mathbb{R}_+$, where z is such that $\phi_i(z) = \rho_i$ for all $i \in \mathcal{I}_+(z)$. When the measures are restricted to $(0, \infty)$, this is the characteristic property of elements of the invariant manifold $\mathcal{M}$, as described in (2.13). Because K only captures the behavior of ξ on $(0, \infty)$, and a general $\xi \in \mathbf{M}^{\mathbf{I}}$ could be such that any of its components has an atom at zero, it follows that $K(\xi) = 0$ if and only if $\xi \in \mathcal{M}^*$, as defined in (3.9).

□

6.2 Smooth Approximation of Measures

We use an approximation argument to prove Theorem 3.1. An approximation argument was also used by Fu and Williams [6]. Consequently, some propositions and proofs are the same as in [6] and we record those results here without proof. We focus on the details that differ from those in [6]. For each positive integer n, let $\varphi_n \in \mathbf{C}_c^\infty(\mathbb{R})$ be such that $\varphi_n \geq 0, \varphi_n(x) = 0$ for all $x \in (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)$, $\varphi_n(x) = \varphi_n(-x)$ for all

$x > 0$, and $\int_{\mathbb{R}} \varphi_n(x)dx = 1$. Given $\xi \in \mathbf{M}$ and $n \in \mathbb{N}$, let ξ^n be the non-negative, absolutely continuous Borel measure on $\mathbb{R}_+$ whose continuous density is given by

$$d_n(x) = \int_{\mathbb{R}_+} \varphi_n(x-y)\xi(dy) = \int_{\mathbb{R}_+} \varphi_n(y-x)\xi(dy) \quad \text{for } x \in \mathbb{R}_+, \tag{6.8}$$

where we have used the symmetry of φ_n for the last equality. Note that $d_n(\cdot)$ is in $\mathbf{C}_b^\infty(\mathbb{R}_+)$, since φ_n is infinitely differentiable with compact support and ξ is a finite measure on $\mathbb{R}_+$. For any bounded, Borel measurable function f defined on $\mathbb{R}_+$, let $(f * \varphi_n)(y) = \int_{\mathbb{R}_+} \varphi_n(y-x)f(x)dx$ for $y \in \mathbb{R}_+$. Then, by Fubini's theorem,

$$\langle f, \xi^n \rangle = \int_{\mathbb{R}_+} f(x) \int_{\mathbb{R}_+} \varphi_n(y-x)\xi(dy)dx = \langle f * \varphi_n, \xi \rangle. \tag{6.9}$$

The next two propositions are the same as Lemmas 9 and 10 in Section 6 of [6], where the first of these is proved by an argument similar to that in Lemma 7.12 of Puha and Williams [23]. We refer the reader to [23, 6] for the proofs noting that they do not rely on whether the fluid model is in the strictly subcritical regime or not.

Proposition 6.4. *Let $\xi \in \mathbf{K} \cap \mathbf{M}_1$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$, we have*

$$\left\langle \mathbb{1}_{(x+\frac{1}{n},\infty)}, \xi \right\rangle \le \langle \mathbb{1}_{(x,\infty)}, \xi^n \rangle \le \left\langle \mathbb{1}_{\left((x-\frac{1}{n})^+,\infty\right)}, \xi \right\rangle, \tag{6.10}$$

$$\langle \chi, \xi \rangle - \frac{\langle \mathbb{1}, \xi \rangle}{n} \le \langle \chi, \xi^n \rangle \le \langle \chi, \xi \rangle + \frac{\langle \mathbb{1}, \xi \rangle}{n}. \tag{6.11}$$

Furthermore, we have $\xi^n \in \mathbf{A}$ for each $n \in \mathbb{N}$ and as $n \to \infty$,

$$\xi^n \xrightarrow{w} \xi \quad \text{and} \quad \langle \chi, \xi^n \rangle \to \langle \chi, \xi \rangle. \tag{6.12}$$

Given a fluid model solution ζ, for each $t \ge 0$ and $i \in \mathcal{I}$, let $\{\zeta_i^n(t)\}_{n=1}^\infty$ be the approximating sequence of measures for $\zeta_i(t)$, as defined above with $\zeta_i(t)$ in place of ξ. Similarly, define ϑ_i^n for each $i \in \mathcal{I}$, $n \in \mathbb{N}$. For any positive integer ℓ, let $\mathcal{C}_{0,\ell} = \{g \in \mathbf{C}_b^1(\mathbb{R}_+) : g = 0 \text{ on } [0, \frac{1}{\ell}]\}$. For $g \in \mathcal{C}_{0,\ell}$ and all $n > \ell$, we have $(g * \varphi_n)(0) = 0$ and $(g * \varphi_n)'(0) = 0$. It follows that $g * \varphi_n \in \mathcal{C}$, where $\mathcal{C}$ is defined in (2.4). For each positive integer n, $i \in \mathcal{I}$, $t \ge 0$ and $x \in \mathbb{R}_+$, let $\vartheta_i^{n,e}$ be the excess lifetime distribution for ϑ_i^n, and

$$\overline{M}_t^{i,n}(x) = \langle \mathbb{1}_{(x,\infty)}, \zeta_i^n(t) \rangle, \qquad \overline{N}_i^n(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^n \rangle, \qquad \overline{N}_i^{n,e}(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^{n,e} \rangle. \tag{6.13}$$

We note that $\vartheta_i^{n,e}$ has density $\overline{N}_i^n(\cdot)/\langle \chi, \vartheta_i^n \rangle$.

The following proposition shows that for all n sufficiently large, $(t,x) \to \overline{M}_t^{i,n}(x)$ satisfies a transport partial differential equation with nonlinear, nonlocal coefficients on intervals of time where $z_i(\cdot)$ is not zero and on intervals for x that are bounded away from zero.

Proposition 6.5. *Assume that ζ is a fluid model solution. Suppose that $i \in \mathcal{I}$ and $0 \le a < b < \infty$ are such that $z_i(t) \ne 0$ for all $t \in [a,b]$. Then, for each positive integer ℓ and all $n > \ell$, $t \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[a,b]$ for each fixed $x \in \mathbb{R}_+$, and $x \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[\frac{1}{\ell}, \infty)$ for each fixed $t \in [a,b]$, and furthermore,*

$$\frac{\partial \overline{M}_t^{i,n}(x)}{\partial t} = \frac{\Lambda_i(t)}{z_i(t)} \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x} + \nu_i \overline{N}_i^n(x), \tag{6.14}$$

for $t \in [a,b]$, $x \ge \frac{1}{\ell}$, where the partial derivatives with respect to time at $t = a, b$ are from the right, left, respectively, and the partial derivative with respect to x at $x = 1/\ell$ is from the right.

Remark 6.1. From (6.8), for each fixed $t \in [0,\infty)$, the measure $\zeta_i^n(t)$ on $\mathbb{R}_+$ has a continuous density function given by

$$m_t^{i,n}(x) = \int_{\mathbb{R}_+} \varphi_n(y-x)\zeta_i(t)(dy) \quad \text{for all } x \in \mathbb{R}_+. \tag{6.15}$$

For any fixed $x \in \mathbb{R}_+$, $t \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x} = -m_t^{i,n}(x)$ (where the derivative at $x = 0$ is from the right) is continuous on $[0, \infty)$, because the fluid model solution ζ_i is continuous as a function of time. It follows that $(t, x) \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x}$ is separately continuous in t and x and hence is jointly measurable on $[0, \infty) \times \mathbb{R}_+$. Via (6.14), this implies joint measurability of $(t, x) \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial t}$ on $[a, b] \times [\frac{1}{\ell}, \infty)$ for any $n > \ell \geq 1$ when $z_i(t) \neq 0$ for all $t \in [a, b]$. Furthermore, from (6.14) and (6.15), we have for any $n > \ell \geq 1$, $t \in [a, b]$ and $x \in \mathbb{R}_+$,

$$\left| \frac{\partial \overline{M}_t^{i,n}(x)}{\partial t} \right| \leq \frac{\Lambda_i(t)}{z_i(t)} \left| -m_t^{i,n}(x) \right| + \nu_i \overline{N}_i^n(x) \tag{6.16}$$

$$\leq \Lambda_i(t) \sup_{y \in \mathbb{R}} \varphi_n(y) + \nu_i \overline{N}_i^n(x). \tag{6.17}$$

It follows that $(t, x) \to \frac{\partial \overline{M}_t^{i,n}(x)}{\partial t}$ is measurable and integrable over the interval $[a, b] \times [\frac{1}{\ell}, \ell]$ for each fixed $n > \ell \geq 1$. These measurability and integrability properties will be needed for a use of Fubini's theorem in the proof of Theorem 3.1 below.

Lemma 6.1. *Suppose that $\vartheta \in \mathbf{K}^\mathbf{I}$ and ζ is a fluid model solution with $\zeta(0) \in \mathbf{K}^\mathbf{I} \cap \mathbf{M}_1^\mathbf{I}$. For any $0 \leq a < b < \infty$, for each $i \in \mathcal{I}$, we have the following uniform bounds:*

$$\sup_{n \in \mathbb{N}} \sup_{t \in [a,b]} \sup_{x \in \mathbb{R}_+} \overline{M}_t^{i,n}(x) \leq \sup_{t \in [a,b]} z_i(t) < \infty, \tag{6.18}$$

$$\sup_{n \in \mathbb{N}} \sup_{t \in [a,b]} \langle \chi, \zeta_i^n(t) \rangle \leq \sup_{t \in [a,b]} (w_i(t) + z_i(t)) < \infty. \tag{6.19}$$

In addition, for each $i \in \mathcal{I}$, as $n \to \infty$, $\zeta_i^n(t) \xrightarrow{w} \zeta_i(t)$, $\vartheta_i^n \xrightarrow{w} \vartheta_i$, $\langle \chi, \vartheta_i^n \rangle \to \langle \chi, \vartheta_i \rangle$, $\vartheta_i^{n,e} \xrightarrow{w} \vartheta_i^e$, $\overline{M}_t^{i,n}(x) \to \overline{M}_t^i(x)$, $\overline{N}_i^n(x) \to \overline{N}_i(x)$ and $\overline{N}_i^{n,e}(x) \to \overline{N}_i^e(x)$ for each $x \in [0, \infty)$.

Proof. By Proposition 2.2 and Remark 2.3, $\zeta(t) \in \mathbf{K}^\mathbf{I} \cap \mathbf{M}_1^\mathbf{I}$ for each $t \geq 0$, and by Proposition 6.4, for each $i \in \mathcal{I}$, $n \in \mathbb{N}, t \geq 0$ and $x \in \mathbb{R}_+$, we have $\overline{M}_t^{i,n}(x) := \langle 1\!\!1_{(x,\infty)}, \zeta_i^n(t) \rangle \leq \overline{M}_t^i\big((x - \frac{1}{n})^+\big) \leq z_i(t)$ and $\langle \chi, \zeta_i^n(t) \rangle \leq w_i(t) + z_i(t)$. Since $z_i(\cdot)$ and $w_i(\cdot)$ are continuous, it follows that for any $0 \leq a < b < \infty$, $\overline{M}_t^{i,n}(x)$ and $\langle \chi, \zeta_i^n(t) \rangle$ have uniform bounds for all $t \in [a, b]$, $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$. Furthermore, by Proposition 6.4, $\zeta_i^n(t) \xrightarrow{w} \zeta_i(t)$, $\vartheta_i^n \xrightarrow{w} \vartheta_i$ and $\langle \chi, \vartheta_i^n \rangle \to \langle \chi, \vartheta_i \rangle$ as $n \to \infty$. It follows, since $\zeta_i(t) \in \mathbf{K}$ and $\vartheta_i \in \mathbf{K}$, that $\overline{M}_t^{i,n}(x) \to \overline{M}_t^i(x)$ and $\overline{N}_i^n(x) \to \overline{N}_i(x)$ for each $x \in \mathbb{R}_+$ as $n \to \infty$, and the density $\overline{N}_i^n(\cdot)/\langle \chi, \vartheta_i^n \rangle$ for $\vartheta_i^{n,e}$ converges everywhere on $\mathbb{R}_+$ to $\overline{N}_i(\cdot)/\langle \chi, \vartheta_i \rangle$, the density for ϑ_i^e, as $n \to \infty$. Since the last sequence of densities is eventually dominated by $2\overline{N}_i\left((\cdot - 1)^+\right)/\langle \chi, \vartheta_i \rangle$, which is integrable on $\mathbb{R}_+$, it follows by dominated convergence that $\vartheta_i^{n,e} \xrightarrow{w} \vartheta_i^e$ as $n \to \infty$, which implies, since $\overline{N}_i^e(\cdot)$ is continuous, that $\overline{N}_i^{n,e}(x) \to \overline{N}_i^e(x)$ as $n \to \infty$ for all $x \in [0, \infty)$. □

The following lemma is used to control $x \to \frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)}$ uniformly in n.

Lemma 6.2. *Suppose Assumption 2 holds and that ζ is a fluid model solution with $\zeta(0) \in \mathbf{K}^\mathbf{I} \cap \mathbf{M}_1^\mathbf{I}$. Let $0 \leq a < b < \infty$ and $i \in \mathcal{I}$ be such that $\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \leq \upsilon_{a,b}$ for all $x \in \mathbb{R}_+$ and $t \in [a, b]$, for some $\upsilon_{a,b} \in (0, \infty)$. Then there is $n_{a,b} \in \mathbb{N}$ (depending only on a, b, C_ϑ and ϑ_i) such that for all $n \geq n_{a,b}$, we have $\frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)} \leq 2\upsilon_{a,b}$ for all $x \in \mathbb{R}_+$ and $t \in [a, b]$.*

Proof. By Proposition 6.4, we have for $t \in [a,b]$, $x \in \mathbb{R}_+$,

$$\begin{aligned}\frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)} = \frac{\langle \mathbb{1}_{(x,\infty)}, \zeta_i^n(t)\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^{n,e}\rangle} &\le \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle \langle \chi, \vartheta_i^n\rangle}{\int_x^\infty \langle \mathbb{1}_{(y,\infty)}, \vartheta_i^n\rangle dy} \\ &\le \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle \langle \chi, \vartheta_i^n\rangle}{\int_{x+\frac{1}{n}}^\infty \langle \mathbb{1}_{(y,\infty)}, \vartheta_i\rangle dy} \\ &= \frac{\langle \chi, \vartheta_i^n\rangle}{\langle \chi, \vartheta_i\rangle} \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle}{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle - \langle \mathbb{1}_{((x-\frac{1}{n})^+,x+\frac{1}{n})}, \vartheta_i^e\rangle} \\ &= \frac{\langle \chi, \vartheta_i^n\rangle}{\langle \chi, \vartheta_i\rangle} \frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \zeta_i(t)\rangle / \langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle}{1 - (\langle \mathbb{1}_{((x-\frac{1}{n})^+,x+\frac{1}{n})}, \vartheta_i^e\rangle / \langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle)}. \qquad (6.20)\end{aligned}$$

Note that $\lim_{n\to\infty} \frac{\langle \chi, \vartheta_i^n\rangle}{\langle \chi, \vartheta_i\rangle} = 1$. For the other term in (6.20), the numerator is bounded above by $\upsilon_{a,b}$ and for the denominator, by Assumption 2, $\frac{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e\rangle} \le C_\vartheta$ for all $x \ge 0$, which implies that

$$\begin{aligned}\frac{\langle \mathbb{1}_{((x-\frac{1}{n})^+,x+\frac{1}{n})}, \vartheta_i^e\rangle}{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle} &= \frac{\int_{(x-\frac{1}{n})^+}^{x+\frac{1}{n}} \frac{\langle \mathbb{1}_{(y,\infty)}, \vartheta_i\rangle}{\langle \chi, \vartheta_i\rangle} dy}{\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle} \\ &\le \frac{2\langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i\rangle}{n\langle \chi, \vartheta_i\rangle \langle \mathbb{1}_{((x-\frac{1}{n})^+,\infty)}, \vartheta_i^e\rangle} \\ &\le \frac{2C_\vartheta}{n\langle \chi, \vartheta_i\rangle}.\end{aligned}$$

Thus for all sufficiently large n (not depending on x), the denominator of the second fraction in the right hand side of (6.20) is greater than 1/2. It follows that for all sufficiently large n (depending only on C_ϑ and ϑ_i),

$$\frac{\overline{M}_t^{i,n}(x)}{\overline{N}_i^{n,e}(x)} \le 2\upsilon_{a,b} \quad \text{for all } t \in [a,b], x \in \mathbb{R}_+.$$

□

6.3 Proof of Theorem 3.1

The following proof is similar to a combination of the proofs of Theorem 1 and Corollary 1 in Section 4 of [6] with $(\rho, 0)$ in place of $(\tilde{\rho}, \delta)$ there. However, since we are now in the critical case, rather than the strictly subcritical case, some aspects in our proof are more delicate and require different justifications due to the more singular form of the Lyapunov function considered here. In addition, our development of conditions for equality to hold everywhere in (3.15) is new.

Proof of Theorem 3.1. Assume that the hypotheses of Theorem 3.1 hold. Since $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ for some $\upsilon > 0$, we have by Lemma 3.1 that $\zeta(t) \in \mathbf{K}_{\upsilon_t^*}^{\mathbf{I}}$ for all $t \ge 0$, where υ_t^* is given in Lemma 3.1. Hence, $\zeta(t) \in \mathbf{K}^{\mathbf{I}} \cap \mathbf{M}_1^{\mathbf{I}}$ for all $t \ge 0$. It follows that for each $i \in \mathcal{I}$ and $t \ge 0$, $x \to \overline{M}_t^i(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to $\langle \chi, \zeta_i(t)\rangle < \infty$) on $\mathbb{R}_+$. Also, under the assumptions on ϑ_i, including Assumption 2, we have that $\vartheta_i \in \mathbf{K}$ and $x \to \overline{N}_i(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to $\langle \chi, \vartheta_i\rangle < \infty$) on $\mathbb{R}_+$.

Fix $i \in \mathcal{I}$. By the upper semicontinuity of $\mathcal{K}_i^\zeta(\cdot)$ (see Lemma 3.6), this function is Borel measurable on $[0,t]$ for each $t \ge 0$. To prove the absolute continuity of $\mathcal{H}_i^\zeta(\cdot)$, it suffices to prove that $\mathcal{K}_i^\zeta(\cdot)$ is integrable over $[0,t]$ and that (3.14) holds, for each $t \ge 0$.

We first prove that if $0 \le a < b < \infty$ such that $z_i(s) \ne 0$ for all $s \in [a,b]$, then $\mathcal{K}_i^\zeta(\cdot)$ is integrable on $[a,b]$ and

$$\mathcal{H}_i^\zeta(b) - \mathcal{H}_i^\zeta(a) = \int_a^b \mathcal{K}_i^\zeta(s) ds. \qquad (6.21)$$

Assuming we have such $a < b$, note that by the last part of Lemma 3.6, $\mathcal{K}_i^\zeta$ is continuous on $[a,b]$ and hence integrable there. To prove that (6.21) holds, recall the form of $\mathcal{K}_i^\zeta(\cdot)$ from (3.12). Using the facts that $\Lambda_i(\cdot) \le \max_j C_j$; $z_i(\cdot)$ is bounded on $[a,b]$, being continuous there; $\frac{\overline{M}_s^i(\cdot)}{\overline{N}_i^e(\cdot)} \le \upsilon_b^*$ for all $s \in [a,b]$, by Lemma 3.1; $\left|\frac{\Lambda_i(\cdot)}{z_i(\cdot)}\right|$ is bounded on $[a,b]$ since $z_i(\cdot)$ is continuous and strictly positive there; and $\int_0^\infty \overline{N}_i(x)dx = \langle \chi, \vartheta_i\rangle = \mu_i^{-1} < \infty$; we see that by dominated convergence,

$$\begin{aligned}\int_a^b \mathcal{K}_i^\zeta(s)ds &= -\frac{\kappa_i}{\rho_i^\alpha}\int_a^b \Lambda_i(s)(z_i(s))^\alpha ds \\ &\quad + \lim_{\ell\to\infty}\frac{\kappa_i}{\rho_i^\alpha}\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right)^\alpha\left(\frac{-\Lambda_i(s)}{z_i(s)}\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\frac{\alpha\overline{N}_i(x)}{\langle\chi,\vartheta_i\rangle} + \nu_i(\alpha+1)\overline{N}_i(x)\right)dxds.\end{aligned} \tag{6.22}$$

Now, for positive integers ℓ and $n > \ell$, by Assumption 2 and since $\zeta(s) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_b^*}$, we have $\vartheta_i \in \mathbf{K}\cap\mathbf{M}_1$ and $\zeta_i(s) \in \mathbf{K}\cap\mathbf{M}_1$ for all $s \in [a,b]$. Then by Lemma 6.1, we have that as $n\to\infty$, $\langle\chi,\vartheta_i^n\rangle \to \langle\chi,\vartheta_i\rangle > 0$, $\overline{N}_i^n(x) \to \overline{N}_i(x)$, $\overline{N}_i^{n,e}(x) \to \overline{N}_i^e(x)$ and $\overline{M}_s^{i,n}(x) \to \overline{M}_s^i(x)$ for each $x \in \mathbb{R}_+$, $s\in[a,b]$. Furthermore, for $s \in [a,b]$, since $\zeta(s) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_b^*}$, we have $\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)} \le \upsilon_b^*$ for all $x \in \mathbb{R}_+$, and then by Lemma 6.2 there is a positive integer $n_{a,b}$ such that for all $n \ge n_{a,b}$, $\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_{i,n}^e(x)} \le 2\upsilon_b^*$ for all $x\in\mathbb{R}_+$ and $s\in[a,b]$. It then follows by the dominated convergence theorem (using the fact from Proposition 6.4 that $\overline{N}_i^n(x) \le \overline{N}_i((x-1)^+$, where the latter is integrable over $x \in [\frac{1}{\ell},\ell]$), that for each fixed positive integer ℓ,

$$\begin{aligned}&\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right)^\alpha\left(\frac{-\Lambda_i(s)}{z_i(s)}\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\frac{\alpha\overline{N}_i(x)}{\langle\chi,\vartheta_i\rangle} + \nu_i(\alpha+1)\overline{N}_i(x)\right)dxds \\ &= \lim_{n\to\infty}\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)}\right)^\alpha\left(\frac{-\Lambda_i(s)}{z_i(s)}\frac{\overline{M}_s^{i.n}(x)}{\overline{N}_i^{n,e}(x)}\frac{\alpha\overline{N}_i^n(x)}{\langle\chi,\vartheta_i^n\rangle} + \nu_i(\alpha+1)\overline{N}_i^n(x)\right)dxds.\end{aligned} \tag{6.23}$$

Using integration by parts on the first term in the integral in (6.23), and the fact that

$$\frac{\partial}{\partial x}\left(\overline{N}_i^{n,e}(x)\right)^{-\alpha} = \frac{\alpha\overline{N}_i^n(x)}{\left(\overline{N}_i^{n,e}(x)\right)^{\alpha+1}\langle\chi,\vartheta_i^n\rangle},$$

the last line in (6.23) is equal to

$$\begin{aligned}&\lim_{n\to\infty}\Bigg(\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)[-(\overline{M}_s^{i,n}(\cdot))^{\alpha+1}(\overline{N}_i^{n,e}(\cdot))^{-\alpha}]_{\frac{1}{\ell}}^{\ell}ds \\ &\quad +(\alpha+1)\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)}\right)^\alpha\left(\frac{\Lambda_i(s)}{z_i(s)}\frac{\partial\overline{M}_s^{i,n}(x)}{\partial x} + \nu_i\overline{N}_i^n(x)\right)dxds\Bigg) \\ &= \lim_{n\to\infty}\Bigg(\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)[-(\overline{M}_s^{i,n}(\cdot))^{\alpha+1}(\overline{N}_i^{n,e}(\cdot))^{-\alpha}]_{\frac{1}{\ell}}^{\ell}ds \\ &\quad +(\alpha+1)\int_a^b\int_{\frac{1}{\ell}}^{\ell}\left(\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)}\right)^\alpha\left(\frac{\partial\overline{M}_s^{i,n}(x)}{\partial s}\right)dxds\Bigg),\end{aligned} \tag{6.24}$$

where we have used Proposition 6.5 for the last equality. By Fubini's theorem, the above is equal to the expression immediately below. For this use of Fubini's theorem, the joint measurability and absolute integrability of the integrand for each fixed $n \ge \max(n_{a,b}, \ell+1)$ follow from Remark 6.1 and the fact that $\frac{\overline{M}_s^{i,n}(x)}{\overline{N}_i^{n,e}(x)} \le \upsilon_b^*$ for all $x \in [\frac{1}{\ell},\ell], s\in[a,b]$.

$$\lim_{n\to\infty}\left(\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)\left[-\frac{(\overline{M}_s^{i,n}(\ell))^{\alpha+1}}{(\overline{N}_i^{n,e}(\ell))^{\alpha}}+\frac{\left(\overline{M}_s^{i,n}\left(\frac{1}{\ell}\right)\right)^{\alpha+1}}{\left(\overline{N}_i^{n,e}\left(\frac{1}{\ell}\right)\right)^{\alpha}}\right]ds\right.$$
$$\left.+\int_{\frac{1}{\ell}}^{\ell}\left((\overline{M}_b^{i,n}(x))^{\alpha+1}-(\overline{M}_a^{i,n}(x))^{\alpha+1}\right)\left(\overline{N}_i^{n,e}(x)\right)^{-\alpha}dx\right)$$
$$=\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)\left[-\frac{(\overline{M}_s^{i}(\ell))^{\alpha+1}}{\left(\overline{N}_i^{e}(\ell)\right)^{\alpha}}+\frac{\left(\overline{M}_s^{i}\left(\frac{1}{\ell}\right)\right)^{\alpha+1}}{\left(\overline{N}_i^{e}\left(\frac{1}{\ell}\right)\right)^{\alpha}}\right]ds \tag{6.25}$$
$$+\int_{\frac{1}{\ell}}^{\ell}\left((\overline{M}_b^{i}(x))^{\alpha+1}-(\overline{M}_a^{i}(x))^{\alpha+1}\right)\left(\overline{N}_i^{e}(x)\right)^{-\alpha}dx,$$

where we have used dominated convergence, provided by Lemmas 6.1 and 6.2, to pass to the limit for the last equality. Note that as $\ell\to\infty$, $\overline{M}_s^i(\ell)\to 0$, $\overline{M}_s^i\left(\frac{1}{\ell}\right)\to z_i(s)$, $\overline{N}_i^e\left(\frac{1}{\ell}\right)\to 1$ and $\overline{M}_s^i(x)\le z_i(s)$, $\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\le \upsilon_b^*$ for all $s\in[a,b], x\in\mathbb{R}_+$. Combining this with the fact that for $s=a,b$, $\left|\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right|\le\upsilon_b^*$ for all $x\in\mathbb{R}_+$ and $x\to\overline{M}_s^i(x)$ is integrable on $\mathbb{R}_+$, we see by dominated convergence that as $\ell\to\infty$, the above expression converges to

$$\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)(z_i(s))^{\alpha+1}ds+\int_0^\infty\left((\overline{M}_b^i(x))^{\alpha+1}-(\overline{M}_a^i(x))^{\alpha+1}\right)\left(\overline{N}_i^e(x)\right)^{-\alpha}dx. \tag{6.26}$$

On substituting the above into (6.22), we obtain

$$\begin{aligned}\int_a^b\mathcal{K}_i^\zeta(s)ds &= -\frac{\kappa_i}{\rho_i^\alpha}\int_a^b\Lambda_i(s)(z_i(s))^\alpha ds+\frac{\kappa_i}{\rho_i^\alpha}\int_a^b\left(\frac{\Lambda_i(s)}{z_i(s)}\right)(z_i(s))^{\alpha+1}ds\\ &\quad+\frac{\kappa_i}{\rho_i^\alpha}\int_0^\infty\left((\overline{M}_b^i(x))^{\alpha+1}-(\overline{M}_a^i(x))^{\alpha+1}\right)\left(\overline{N}_i^e(x)\right)^{-\alpha}dx\\ &= \mathcal{H}_i^\zeta(b)-\mathcal{H}_i^\zeta(a),\end{aligned} \tag{6.27}$$

as desired.

We now turn to proving that $\mathcal{K}_i^\zeta(\cdot)$ is integrable over $[0,t]$ and 3.14) holds for each $t\ge 0$. This clearly holds for $t=0$, so we consider $t>0$ fixed. If $z_i(s)\neq 0$ for all $s\in[0,t]$, then the result follows immediately from what we proved for (6.21) with $a=0$ and $b=t$. So we only need to treat the case where $z_i(s)=0$ for some $s\in[0,t]$. Assuming this, let $s^*=\inf\{s\in[0,t]: z_i(s)=0\}$ and $t^*=\sup\{s\in[0,t]:z_i(s)=0\}$. Then, $0\le s^*\le t^*\le t$, $z_i(s^*)=z_i(t^*)=0$ and $z_i(s)>0$ for $s\in(0,s^*)\cup(t^*,t)$. (Note that the interval $(0,s^*)$ is empty if $z_i(0)=0$ and (t^*,t) is empty if $z_i(t)=0$.) In any event, we can write the open set $\mathcal{T}_t^i=\{s\in(0,t):z_i(s)>0\}$ as a (finite or countable) union of disjoint open intervals:

$$\mathcal{T}_t^i=(0,s^*)\cup\left(\bigcup_n(s_n,t_n)\right)\cup(t^*,t), \tag{6.28}$$

where $\bigcup_n(s_n,t_n)\subset(s^*,t^*)$ and $z_i(s_n)=z_i(t_n)=0$ for each n.

Recall the definitions of $\mathrm{k}_i^{(1)},\mathrm{k}_i^{(2)},\mathrm{k}_i^{(3)}$ from the proof of Lemma 3.4. For all $s\ge 0$, let

$$\begin{aligned}k_i^{(1)}(s)&=\mathrm{k}_i^{(1)}(\zeta(s))=-\kappa_i\rho_i^{-\alpha}\Lambda_i(s)\big(z_i(s)\big)^\alpha,\\ k_i^{(2)}(s)&=\mathrm{k}_i^{(2)}(\zeta(s))=-\kappa_i\rho_i^{-\alpha}\int_0^\infty\left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right)^{\alpha+1}\overline{N}_i(x)\frac{\alpha\Lambda_i(s)}{z_i(s)\langle\chi,\vartheta_i\rangle}\mathbb{1}_{(0,\infty)}\big(z_i(s)\big)dx,\\ k_i^{(3)}(s)&=\mathrm{k}_i^{(3)}(\zeta(s))=\kappa_i\rho_i^{-\alpha}\nu_i(\alpha+1)\int_0^\infty\left(\frac{\overline{M}_s^i(x)}{\overline{N}_i^e(x)}\right)^{\alpha}\overline{N}_i(x)\mathbb{1}_{(0,\infty)}\big(z_i(s)\big)dx.\end{aligned} \tag{6.29}$$

Then we have $|k_i^{(1)}(s)| \le \kappa_i \rho_i^{-\alpha}(\max_{j\in\mathcal{J}} C_j)\sup_{s\in[0,t]}(z_i(s))^\alpha$ for $s \in [0,t]$, which implies that $\int_{[0,t]} |k_i^{(1)}(s)|ds < \infty$. By Lemma 3.1, $|k_i^{(3)}(s)| \le \kappa_i \rho_i^{-\alpha}(\alpha+1)\nu_i(\upsilon_t^*)^\alpha \langle \chi, \vartheta_i \rangle$ for $s \in [0,t]$, which implies that $\int_{[0,t]} |k_i^{(3)}(s)|ds < \infty$.

For each fixed n, equation (6.21) gives that for any $[a,b] \subset (s_n, t_n)$

$$\int_{[a,b]} k_i^{(1)}(s)ds + \int_{[a,b]} k_i^{(2)}(s)ds + \int_{[a,b]} k_i^{(3)}(s)ds = \mathcal{H}_i^\zeta(b) - \mathcal{H}_i^\zeta(a).$$

Thus,

$$-\int_{[a,b]} k_i^{(2)}(s)ds = \int_{[a,b]} k_i^{(1)}(s)ds + \int_{[a,b]} k_i^{(3)}(s)ds + \mathcal{H}_i^\zeta(a) - \mathcal{H}_i^\zeta(b).$$

By the continuity of $\mathcal{H}_i^\zeta(\cdot)$ established in Lemma 3.3, as $a \to s_n$ and $b \to t_n$, $\mathcal{H}_i^\zeta(b) \to \mathcal{H}_i^\zeta(t_n) = 0$ and $\mathcal{H}_i^\zeta(a) \to \mathcal{H}_i^\zeta(s_n) = 0$. It follows from the above and since $k_2(s) \le 0$ for all $s \ge 0$, that

$$\begin{aligned}\int_{(s_n,t_n)} |k_2(s)|ds &= -\int_{(s_n,t_n)} k_i^{(2)}(s)ds &(6.30)\\ &= \int_{(s_n,t_n)} k_i^{(1)}(s)ds + \int_{(s_n,t_n)} k_i^{(3)}(s)ds + \mathcal{H}_i^\zeta(s_n) - \mathcal{H}_i^\zeta(t_n) &(6.31)\\ &= \int_{(s_n,t_n)} k_i^{(1)}(s)ds + \int_{(s_n,t_n)} k_i^{(3)}(s)ds. &(6.32)\end{aligned}$$

In a similar manner, we can obtain

$$\int_{(0,s^*)} |k_2(s)|ds = -\int_{(0,s^*)} k_i^{(2)}(s)ds = \int_{(0,s^*)} k_i^{(1)}(s)ds + \int_{(0,s^*)} k_i^{(3)}(s)ds + \mathcal{H}_i^\zeta(0), \tag{6.33}$$

since $\mathcal{H}_i^\zeta(s^*) = 0$, and

$$\int_{(t^*,t)} |k_2(s)|ds = -\int_{(t^*,t)} k_i^{(2)}(s)ds = \int_{(t^*,t)} k_i^{(1)}(s)ds + \int_{(t^*,t)} k_i^{(3)}(s)ds - \mathcal{H}_i^\zeta(t). \tag{6.34}$$

since $\mathcal{H}_i^\zeta(t^*) = 0$. Hence using the integrability of $k_i^{(1)}$ and $k_i^{(3)}$ on $[0,t]$, the fact that $k_i^{(2)}$ is zero on $(0,t)\setminus\mathcal{T}_t^i$ and non-positive on $\mathcal{T}_t^i$, together with the disjointness of the intervals in the representation (6.28) for $\mathcal{T}_t^i$, we have $\int_{(0,t)} |k_i^{(2)}(s)|ds < \infty$. Thus, $\mathcal{K}_i^\zeta = k_i^{(1)} + k_i^{(2)} + k_i^{(3)}$ is integrable on $(0,t)$, and by (6.32)–(6.34), we have

$$\int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds = 0 \quad \text{for each } n, \tag{6.35}$$

$$\int_{(0,s^*)} \mathcal{K}_i^\zeta(s)ds = -\mathcal{H}_i^\zeta(0) \quad \text{and} \quad \int_{(t^*,t)} \mathcal{K}_i^\zeta(s)ds = \mathcal{H}_i^\zeta(t). \tag{6.36}$$

Combining all of the above, and using the integrability of $\mathcal{K}_i^\zeta$ on $[0,t]$, the fact that $\mathcal{K}_i^\zeta(\cdot)$ is zero on $(0,t)\setminus\mathcal{T}_t^i$ and the disjointness of the intervals in the representation (6.28), we have

$$\begin{aligned}\int_0^t \mathcal{K}_i^\zeta(s)ds &= \int_{(0,s^*)} \mathcal{K}_i^\zeta(s)ds + \sum_n \int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds + \int_{(t^*,t)} \mathcal{K}_i^\zeta(s)ds\\ &= -\mathcal{H}_i^\zeta(0) + 0 + \mathcal{H}_i^\zeta(t),\end{aligned}$$

which is the desired result (3.14).

The inequality (3.15) follows immediately from Lemma 3.5 with $\xi = \zeta(t)$ where $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_t^*} \subset \mathbf{M}^{\mathbf{I}}_{\upsilon_t^*}$. By Lemma 3.5, equality holds everywhere in (3.15) if and only if $\zeta(t) \in \mathcal{M}^*$. Furthermore, since $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon_t^*}$, its components cannot have atoms at zero and so the $\mathcal{M}^*$ can be replaced by $\mathcal{M}$ in this "if and only if statement" just stated. The non-positivity of $\mathcal{K}^\zeta(\cdot)$ yields the non-increasing property of $\mathcal{H}^\zeta(\cdot)$, and the fact from that $\mathcal{K}^\zeta(t) < 0$ at times $t \in [0,\infty)$ where $\zeta(t) \notin \mathcal{M}$ yields that $\mathcal{H}^\zeta(\cdot)$ is strictly decreasing at such times. □

7 Properties of Workload, H, $\underline{F}$, G, and Total Mass for Fluid Model Solutions

In this section, we develop some properties of fluid model solutions and the relationship between H and $\underline{F}$ that will be needed for the proofs of our main results.

7.1 Properties of Workload

Lemma 7.1. *Suppose Assumption 1 holds and ζ is a fluid model solution satisfying $w_i(0) < \infty$ for all $i \in \mathcal{I}$. Then $t \to \widetilde{w}_j(\zeta(t))$ is a non-decreasing function on $[0, \infty)$ for each $j \in \mathcal{J}_*$.*

Proof. For $j \in \mathcal{J}_*$, by (2.6), Definition 2.1 and Assumption 1, we have for each $t \geq 0$,

$$\begin{aligned}
\widetilde{w}_j(\zeta(t)) &= \sum_{i \in \mathcal{I}} R_{ji} w_i(t) \\
&= \sum_{i \in \mathcal{I}} R_{ji} \Big(w_i(0) + \int_0^t \big(\rho_i - \Lambda_i(s)\big) \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds \Big) \\
&= \widetilde{w}_j(\zeta(0)) + \sum_{i \in \mathcal{I}} R_{ji} \rho_i t - \sum_{i \in \mathcal{I}} R_{ji} \tau_i(t) \\
&= \widetilde{w}_j(\zeta(0)) + u_j(t).
\end{aligned}$$

The desired result follows from the fact that $u_j(\cdot)$ is non-decreasing, by Definition 2.2(ii) for a fluid model solution. □

Lemma 7.2. *Suppose Assumptions 1 and 2 hold. Let $\upsilon > 0$. Then, for any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, we have*

$$\sup_{t \geq 0} \max_{i \in \mathcal{I}} w_i(t) \leq B_\upsilon, \tag{7.1}$$

where B_υ is a finite, positive constant depending only on $\upsilon, \alpha, \rho, \kappa, \langle \chi, \vartheta^e \rangle$.

Proof. Fix $i \in \mathcal{I}$ and $t \geq 0$. Then

$$\begin{aligned}
w_i(t) &= \int_0^\infty \overline{M}_t^i(x) dx \\
&= \langle \chi, \vartheta_i^e \rangle \int_0^\infty \frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \frac{\overline{N}_i^e(x)}{\langle \chi, \vartheta_i^e \rangle} dx \\
&\leq \langle \chi, \vartheta_i^e \rangle \Bigg(\int_0^\infty \left(\frac{\overline{M}_t^i(x)}{\overline{N}_i^e(x)} \right)^{\alpha+1} \frac{\overline{N}_i^e(x)}{\langle \chi, \vartheta_i^e \rangle} dx \Bigg)^{\frac{1}{\alpha+1}},
\end{aligned} \tag{7.2}$$

where the last inequality follows from Jensen's inequality, since $\frac{\overline{N}_i^e(\cdot)}{\langle \chi, \vartheta_i^e \rangle}$ is a probability density (for the probability measure $(\vartheta_i^e)^e$). We observe that the last line in (7.2) equals

$$\left(\frac{\rho_i^\alpha \mathcal{H}_i^\zeta(t) \langle \chi, \vartheta_i^e \rangle^\alpha}{\kappa_i} \right)^{\frac{1}{\alpha+1}}. \tag{7.3}$$

By the definition of $\mathcal{H}^\zeta(\cdot)$, $\mathcal{H}_i^\zeta(t) \leq (\alpha+1)\mathcal{H}^\zeta(t)$. By Theorem 3.1, $\mathcal{H}^\zeta(\cdot)$ is non-increasing and so $\mathcal{H}^\zeta(t)$ is bounded above by $\mathcal{H}^\zeta(0)$ and hence (7.3) is bounded above by

$$\left(\frac{\rho_i^\alpha (\alpha+1) \mathcal{H}^\zeta(0) \langle \chi, \vartheta_i^e \rangle^\alpha}{\kappa_i} \right)^{\frac{1}{\alpha+1}} \leq \left(\frac{\rho_i^\alpha}{\kappa_i} \langle \chi, \vartheta_i^e \rangle^\alpha \upsilon^{\alpha+1} \sum_{k \in \mathcal{I}} \frac{\kappa_k \langle \chi, \vartheta_k^e \rangle}{\rho_k^\alpha} \right)^{\frac{1}{\alpha+1}},$$

where we have used the fact that $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ for the last inequality. The desired result follows because $i \in \mathcal{I}$ and $t \geq 0$ were arbitrary and by taking the maximum over $i \in \mathcal{I}$. □

7.2 Relationship between H and $\underline{F}$

Lemma 7.3. *Let $\xi \in \cup_{\upsilon>0}\mathbf{M}^{\mathbf{I}}_{\upsilon}$. Then*

$$\underline{F}\,(\widetilde{w}(\xi)) \le F(\check{z}) \le H(\xi), \tag{7.4}$$

where $\check{z}_i = \frac{\langle \chi, \xi_i\rangle}{\langle \chi, \vartheta^e_i\rangle}$ for $i \in \mathcal{I}$ and $\widetilde{w}_j(\xi) = \sum_{i\in\mathcal{I}} R_{ji}\langle \chi, \xi_i\rangle$ for $j \in \mathcal{J}_*$. If, in addition, Assumption 1 holds, then the inequalities in (7.4) are all equalities if and only if $\xi \in \mathcal{M}^*$.

Proof. We have

$$\begin{aligned} H(\xi) &= \sum_{i\in\mathcal{I}} \frac{\kappa_i\langle \chi, \vartheta^e_i\rangle}{(\alpha+1)\rho_i^\alpha} \int_0^\infty \left(\frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta^e_i\rangle}\right)^{\alpha+1} \frac{\overline{N}^e_i(x)}{\langle \chi, \vartheta^e_i\rangle} dx \\ &\ge \sum_{i\in\mathcal{I}} \frac{\kappa_i\langle \chi, \vartheta^e_i\rangle}{(\alpha+1)\rho_i^\alpha} \left(\int_0^\infty \frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta^e_i\rangle} \frac{\overline{N}^e_i(x)}{\langle \chi, \vartheta^e_i\rangle} dx\right)^{\alpha+1} \quad \text{by Jensen's Inequality} \qquad (7.5) \\ &= \sum_{i\in\mathcal{I}} \frac{\kappa_i\langle \chi, \vartheta^e_i\rangle}{(\alpha+1)\rho_i^\alpha} \left(\frac{\langle \chi, \xi_i\rangle}{\langle \chi, \vartheta^e_i\rangle}\right)^{\alpha+1} \\ &= F(\check{z}) \\ &\ge \underline{F}(\widetilde{w}(\xi)), \qquad (7.6) \end{aligned}$$

where the last inequality follows because $\check{w}(\check{z}) = \widetilde{w}(\xi)$ for $\check{w}(\cdot)$ defined as in Lemma 3.7, and so $\check{z}$ is feasible for the optimization problem (3.16) for which $\underline{F}(\widetilde{w}(\xi))$ is the optimal value. The stream of inequalities above establishes (7.4).

We now assume that Assumption 1 holds and characterize when equality holds everywhere in (7.4). By the sharp version of Jensen's inequality, equality holds in (7.5) if and only if $x \to \frac{\langle \mathbb{1}_{(x,\infty)}, \xi_i\rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta^e_i\rangle}$ is a constant for $x \in \mathbb{R}_+$ such that $\overline{N}^e_i(x) \ne 0$. It follows that equality holds in (7.5) if and only if for each $i \in \mathcal{I}$, $\xi_i = a_i\delta_0 + b_i\vartheta^e_i$ for some $a_i, b_i \in [0,\infty)$. For ξ of this form, $\check{z} = b = (b_1, \dots, b_{\mathbf{I}})$. The inequality in (7.6) is an equality if and only if $\check{z}$ is the optimal solution for the optimization problem (3.16) with $\widetilde{w} = \widetilde{w}(\xi) = \check{w}(\check{z})$, i.e., $\check{z} = \Delta(\check{w}(\check{z}))$. It then follows from Lemma 3.7, which requires Assumption 1, that the inequality in (7.6) is an equality if and only if $\check{z} \in \mathcal{P}$. Hence, by the definition of $\mathcal{M}^*$, both inequalities in (7.4) are equalities if and only if $\xi \in \mathcal{M}^*$, as defined in (3.9). □

7.3 Properties of G: Proof of Lemma 4.1

Proof of Lemma 4.1. For (i), fix $\upsilon > 0$. If $\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon}$, then $H(\xi)$ is finite and $w_i(\xi) \le \upsilon\langle \chi, \vartheta^e_i\rangle < \infty$ for all $i \in \mathcal{I}$, and so $\widetilde{w}_j(\xi) < \infty$ for each $j \in \mathcal{J}_*$, since $|\mathcal{I}|$ is finite. It follows that $G(\xi)$ is well defined and finite. By Lemma 7.3, $G(\xi) \ge 0$. For the continuity, suppose $\{\xi_n\}_{n\in\mathbb{N}}, \xi$ are in $\mathbf{M}^{\mathbf{I}}_{\upsilon}$ and $\xi_n \xrightarrow{w} \xi$ as $n \to \infty$. Then $\langle \mathbb{1}_{(x,\infty)}, \xi_n\rangle \to \langle \mathbb{1}_{(x,\infty)}, \xi\rangle$ for almost every $x \in \mathbb{R}_+$ where $\langle \mathbb{1}_{(x,\infty)}, \xi_n\rangle \le \upsilon\langle \mathbb{1}_{(x,\infty)}, \vartheta^e_i\rangle$ for each n and x, and so it follows by dominated convergence that $w_i(\xi_n) = \int_0^\infty \langle \mathbb{1}_{(x,\infty)}, \xi_n\rangle dx \to w_i(\xi) = \int_0^\infty \langle \mathbb{1}_{(x,\infty)}, \xi\rangle dx$ and $\widetilde{w}_j(\xi_n) \to \widetilde{w}_j(\xi)$ as $n \to \infty$ for each $i \in \mathcal{I}, j \in \mathcal{J}_*$. It then follows from the continuity of H on $\mathbf{M}^{\mathbf{I}}_{\upsilon}$ (see Lemma 3.2) and of $\underline{F}$ on $\mathbb{R}^{\mathbf{J}_*}_+$ (see Proposition 3.1) that $G(\xi_n) \to G(\xi)$ as $n \to \infty$. Hence G is continuous on $\mathbf{M}^{\mathbf{I}}_{\upsilon}$.

For (ii), assume that Assumption 1 holds and suppose that $\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon}$ for some $\upsilon > 0$. Noting that $G(\xi) = 0$ if and only if equality holds everywhere in (7.4), we conclude from the last part of Lemma 7.3 (which assumes that Assumption 1 holds) that $G(\xi) = 0$ if and only if $\xi \in \mathcal{M}^*$. □

7.4 Property of Total Mass

The next lemma is an important element in our proof of the convergence of fluid model solutions to the invariant manifold. The proof, in part, uses some ideas from the proof of Lemma 5.1 in the paper of Puha and Williams [23] for a critical fluid model of a single class processor sharing queue. However, the proof given here also has new elements needed to treat general bandwidth sharing policies, which allocate bandwidth to routes in a utility-based, state-dependent manner, whereas for the single class processor sharing queue situation treated in [23], the bandwidth allocated to the class is always one.

Lemma 7.4. *Suppose Assumptions 1 and 2 hold. Fix $\upsilon > 0$. Then, for any fluid model solution ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$, we have*

$$\sup_{t\geq 0}\max_{i\in\mathcal{I}} z_i(t) \leq \widetilde{B}_{\upsilon}, \tag{7.7}$$

where $z_i(t) = \langle \mathbb{1}, \zeta_i(t)\rangle$, $i \in \mathcal{I}$, $t \geq 0$, and $\widetilde{B}_{\upsilon}$ is a finite, positive constant depending only on $\upsilon, \alpha, \nu, \rho, C, \kappa, \langle \chi, \vartheta^e\rangle$.

Proof. It is apparent from the form of the objective function in the optimization problem (2.3) that we have the scaling property:

$$\phi_i(rz) = \phi_i(z) \quad \text{for all } i \in \mathcal{I}, z \in \mathbb{R}^{\mathbf{I}}_{+} \text{ and } r > 0. \tag{7.8}$$

Fix $\upsilon > 0$. Consider a fluid model solution ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_{\upsilon}$. By Lemma 7.2, we know that

$$\langle \chi, \zeta_i(t)\rangle \leq B_{\upsilon} \quad \text{for all } t \geq 0,\ i \in \mathcal{I}. \tag{7.9}$$

Let $\nu^* = \max_{i\in\mathcal{I}} \nu_i$ and

$$\gamma = \min_{i\in\mathcal{I}} \min\left\{\phi_i(z) : z \in \mathbb{R}^{\mathbf{I}}_{+}, z_i \geq \frac{1}{4},\ z_k \leq \frac{3}{2} \text{ for all } k \in \mathcal{I}\right\}. \tag{7.10}$$

We note from the properties of ϕ described in Remark 2.1 that, for each $i \in \mathcal{I}$, ϕ_i is continuous and strictly positive on the compact set

$$\left\{z \in \mathbb{R}^{\mathbf{I}}_{+} : z_i \geq \frac{1}{4}, z_k \leq \frac{3}{2} \text{ for all } k \in \mathcal{I}\right\},$$

and so $\gamma > 0$. Furthermore, from the scaling property (7.8) of ϕ, we have that for each $a > 0$,

$$\gamma = \min_{i\in\mathcal{I}} \min\left\{\phi_i(z) : z \in \mathbb{R}^{\mathbf{I}}_{+}, z_i \geq \frac{a}{4},\ z_k \leq \frac{3a}{2} \text{ for all } k \in \mathcal{I}\right\}. \tag{7.11}$$

Let $\beta = \frac{B_{\upsilon}\nu^*}{\gamma}$ and $f(x) = x^2 - 6\beta x + \beta^2$. The quadratic function f has two roots, the largest of which is $x^* = \beta(3 + 2\sqrt{2})$, and so $f(x) \geq 0$ for $x \geq x^*$. Let $a^* = \max(\upsilon, x^*)$, $\ell = (a^* - \beta)/2\nu^*$, and $b^* = a^* + \nu^*\ell$. Then $f(a^*) \geq 0$, $\ell > 0$, $\nu^*\ell \leq \frac{a^*}{2}$, and $b^* \leq \frac{3a^*}{2}$.

We shall prove the following: for $n = 0, 1, 2, \ldots$, for each $i \in \mathcal{I}$,

$$z_i(n\ell) \leq a^* \quad \text{and} \tag{7.12}$$

$$z_i(t) \leq b^* \quad \text{for all } t \in [n\ell, (n+1)\ell]. \tag{7.13}$$

Once this is proved, we obtain that

$$\sup_{t\in[0,\infty)}\sup_{i\in\mathcal{I}} z_i(t) \leq b^*, \tag{7.14}$$

and the desired result holds with $\tilde{B}_{\upsilon} = b^*$.

We shall prove (7.12)–(7.13) by induction. Before commencing that proof, we first prove some preliminary estimates that hold for all $n = 0, 1, 2, \ldots$. For this, fix $n \in \{0, 1, 2, \ldots\}$ and $i \in \mathcal{I}$. We consider two cases:

(I) $z_i(s) \neq 0$ for all $s \in [n\ell, (n+1)\ell]$,

(II) $z_i(s) = 0$ for some $s \in [n\ell, (n+1)\ell]$.

In case (I), by Proposition 2.3, on setting $x = 0$ in (2.15), we have for $t \in [n\ell, (n+1)\ell]$,

$$\begin{aligned}
z_i(t) = \overline{M}^i_t(0) &= \overline{M}^i_{n\ell}(S^i_{n\ell,t}) + \nu_i \int_{n\ell}^t \overline{N}_i(S^i_{u,t})du \\
&\leq \overline{M}^i_{n\ell}(S^i_{n\ell,t}) + \nu^*\ell && (7.15)\\
&\leq \frac{\int_0^{S^i_{n\ell,t}} \overline{M}^i_{n\ell}(x)dx}{S^i_{n\ell,t}} + \nu^*\ell, && (7.16)\\
&\leq \frac{w_i(n\ell)}{S^i_{n\ell,t}} + \nu^*\ell, && (7.17)
\end{aligned}$$

where we used the non-increasing property of $\overline{M}^i_{n\ell}(\cdot)$ for the inequality in (7.16). Setting $t = (n+1)\ell$ in (7.17), we obtain

$$z_i((n+1)\ell) \leq \frac{w_i(n\ell)}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell \leq \frac{B_\upsilon}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell, \tag{7.18}$$

where we used Lemma 7.2 for the last inequality.

Thus, in case (I), if $z_i(n\ell) \leq a^*$, then by (7.15), since $\overline{M}^i_{n\ell}(\cdot)$ is non-increasing, we have for all $t \in [n\ell,(n+1)\ell]$:

$$z_i(t) \leq \overline{M}^i_{n\ell}(0) + \nu^*\ell \leq a^* + \nu^*\ell = b^*. \tag{7.19}$$

Hence, we see that in case (I), (7.13) follows once (7.12) is proved.

In case (II), by Proposition 2.3, for $t \in [n\ell, s_0)$ where $s_0 = \inf\{s \geq n\ell : z_i(s) = 0\}$, if $z_i(n\ell) \leq a^*$, then

$$\begin{aligned} z_i(t) &\leq z_i(n\ell) + \nu^*\ell && (7.20)\\ &\leq a^* + \nu^*\ell = b^*, && (7.21) \end{aligned}$$

and for any $t \in [s_0,(n+1)\ell]$, either $z_i(t) = 0$ or $z_i(t) > 0$ and by Remark 2.8, for $s_t = \sup\{s \in [n\ell, t) : z_i(s) = 0\}$, we have

$$\begin{aligned} z_i(t) &= \nu_i \int_{s_t}^t \overline{N}_i(x + S^i_{u,t})du \\ &\leq \nu^*\ell \\ &\leq b^*. && (7.22) \end{aligned}$$

Thus, in case (II), if $z_i(n\ell) \leq a^*$, then $z_i(t) \leq b^*$ for all $t \in [n\ell,(n+1)\ell]$.

Combining all of the above, we see that in either case (I) or case (II), (7.13) follows once (7.12) is proved. Also, in case (II),

$$\begin{aligned} z_i((n+1)\ell) &\leq \nu_i \int_{s_{(n+1)\ell}}^{(n+1)\ell} \overline{N}_i(x + S^i_{u,t})du \\ &\leq \nu^*\ell \\ &\leq \frac{a^*}{2}. && (7.23) \end{aligned}$$

We now proceed to the induction proof. Consider first the case of $n = 0$. Fix $i \in \mathcal{I}$. Then by the definition of a^*, $z_i(0) \leq \upsilon \leq a^*$, and from the consideration of cases (I) and (II) above, it follows that $z_i(t) \leq b^*$ for all $t \in [0,\ell]$. Thus, (7.12) and (7.13) hold for $n = 0$ and since $i \in \mathcal{I}$ was arbitrary, they hold for all $i \in \mathcal{I}$ for $n = 0$.

Suppose now for the induction step that (7.12) and (7.13) hold for some $n \geq 0$ for all $i \in \mathcal{I}$. We desire to prove that these inequalities hold with $n+1$ in place of n for all $i \in \mathcal{I}$. For this, fix $i \in \mathcal{I}$. By the consideration of cases (I) and (II) above, we know that it suffices to prove (7.12) holds with $n+1$ in place of n, since (7.13) follows once (7.12) is proved with $n+1$ in place of n. We consider two cases:

(i) $z_i(s) < \frac{a^*}{4}$ for some $s \in [n\ell,(n+1)\ell]$,

(ii) $z_i(s) \geq \frac{a^*}{4}$ for all $s \in [n\ell,(n+1)\ell]$.

Consider case (i) first. If $z_i(s) = 0$ for some $s \in [n\ell,(n+1)\ell]$, then we are in case (II) and by (7.23), we have that $z_i((n+1)\ell) \leq a^*/2 < a^*$ and then (7.12) holds. On the other hand, if $z_i(s) \neq 0$ for all $s \in [n\ell,(n+1)\ell]$, then by Proposition 2.3 we have

$$\begin{aligned} z_i((n+1)\ell) &\leq \overline{M}^i_{(n+1)\ell}(0) = \overline{M}^i_{t_n}(S^i_{t_n,(n+1)\ell}) + \nu_i \int_{t_n}^{(n+1)\ell} \overline{N}_i(S^i_{u,(n+1)\ell})du \\ &\leq z_i(t_n) + \nu^*\ell \\ &\leq \frac{a^*}{4} + \frac{a^*}{2} \\ &< a^*, && (7.24) \end{aligned}$$

where $t_n = \inf\{s \geq n\ell : z_i(s) \leq \frac{a^*}{4}\}$. Thus, (7.12) holds with $n+1$ in place of n in case (i).

Now we suppose that we are in case (ii). Then, we are also in case (I), and by (7.18) we have that

$$z_i((n+1)\ell) \quad \leq \quad \frac{w_i(n\ell)}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell \leq \frac{B_\upsilon}{S^i_{n\ell,(n+1)\ell}} + \nu^*\ell, \tag{7.25}$$

where

$$S^i_{n\ell,(n+1)\ell} \quad = \quad \int_{n\ell}^{(n+1)\ell} \frac{\phi_i(z(s))}{z_i(s)} ds \geq \frac{\gamma\ell}{b^*}, \tag{7.26}$$

and we have used the property (7.11) of γ with $a = a^*$, and the facts that for all $s \in [n\ell, (n+1)\ell]$, $z_i(s) \geq \frac{a^*}{4}$ (since we are in case (ii)), and $z_k(s) \leq b^* \leq \frac{3a^*}{2}$ for all $k \in \mathcal{I}$ (since (7.13) holds with arbitrary k in place of i, by the induction assumption). Combining (7.25) with (7.26), we obtain

$$\begin{aligned} z_i((n+1)\ell) \quad &\leq \quad \frac{B_\upsilon b^*}{\gamma\ell} + \nu^*\ell \\ &= \quad \frac{1}{4\nu^*\ell}\left(4\beta(a^* + \nu^*\ell) + (2\nu^*\ell)^2\right) \\ &= \quad \frac{1}{2(a^*-\beta)}\left(4\beta a^* + 2\beta(a^*-\beta) + (a^*-\beta)^2\right) \\ &= \quad \frac{1}{2(a^*-\beta)}\left((a^*)^2 + 4\beta a^* - \beta^2\right) \end{aligned} \tag{7.27}$$

where we substituted for b^* and used the definition of β for the second line, substituted for ℓ for the third line, and simplified the expression for the last line. The expression on the right hand side of the inequality in (7.27) is less than or equal to a^* if and only if

$$(a^*)^2 - 6\beta a^* + \beta^2 \geq 0. \tag{7.28}$$

The left hand side of (7.28) is $f(a^*)$ and it follows from the fact $a^* \geq x^*$, the largest root of the quadratic f, that (7.28) holds. It follows that we must have $z_i((n+1)\ell) \leq a^*$. This concludes the proof that (7.12) holds with $n+1$ in place of n in case (ii).

Combining all of the preceding arguments, and using the fact that $i \in \mathcal{I}$ was arbitrary, we see that (7.12) (and hence (7.13)) holds with $n+1$ in place of n for all $i \in \mathcal{I}$. This completes the induction step and hence (7.12) and (7.13) hold for all $i \in \mathcal{I}$ and $n = 0, 1, 2, \ldots$. □

8 Proofs of Main Results: Theorems 5.1, 5.2 and 5.3

8.1 Proof of Theorem 5.1

Proof of Theorem 5.1. Property (i) follows by combining Lemma 3.1 with Lemma 4.1 and the continuity of $\zeta(\cdot)$ on $[0,\infty)$. (We note that this part uses Assumption 2, but does not need Assumption 1.) For property (ii), for $t \geq 0$, by Lemma 3.1 and (ii) of Lemma 4.1, $\mathcal{G}^\zeta(t) = 0$ if and only $\zeta(t) \in \mathcal{M}^*$. Furthermore, by Lemma 3.1, $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon^*_t}$ and so it has no atoms (including no atom at zero). It follows that $\mathcal{M}^*$ can be replaced by $\mathcal{M}$ in the "if and only if" statement. Hence, property (ii) holds. For property (iii), by Theorem 3.1, $\mathcal{H}^\zeta(\cdot)$ is non-increasing. Furthermore, $\underline{F}(\widetilde{w}(\zeta(\cdot)))$ is non-decreasing by Proposition 3.1 and Lemma 7.1. Hence $\mathcal{G}^\zeta(\cdot)$ is non-increasing. In addition, by Theorem 3.1, $\mathcal{H}^\zeta(\cdot)$ is strictly decreasing at all $t \geq 0$ such that $\zeta(t) \notin \mathcal{M}$, which implies $\mathcal{G}^\zeta(\cdot)$ is strictly decreasing at all $t \geq 0$ such that $\zeta(t) \notin \mathcal{M}$. □

8.2 Fluid Model Solutions Stay in Relatively Compact Sets

The next lemma provides a key step in the proof that fluid model solutions stay in certain relatively compact sets.

Lemma 8.1. *Suppose Assumptions 1 and 2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ and any $t \geq 0$, let $z^\zeta(t) = \langle \mathbb{1}, \zeta(t) \rangle$. Define*

$$M_\upsilon = \sup \left\{ \frac{z_i^\zeta(t)}{\phi_i(z^\zeta(t))} \mathbb{1}_{\{z_i^\zeta(t) \neq 0\}} : i \in \mathcal{I}, t \geq 0, \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}} \right\}. \tag{8.1}$$

Then $M_\upsilon < \infty$.

Proof. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, for any $t \geq 0$ and $i \in \mathcal{I}$ such that $z_i^\zeta(t) > 0$, by Proposition 3.2, we have

$$\frac{z_i^\zeta(t)}{\phi_i(z^\zeta(t))} = \left(\frac{\sum_{j \in \mathcal{J}} p_j^\zeta(t) R_{ji}}{\kappa_i} \right)^{\frac{1}{\alpha}} \tag{8.2}$$

where $p^\zeta(t) \in \mathbb{R}_+^{\mathbf{J}}$ satisfies conditions (3.17)–(3.20) with $p = p^\zeta(t), z = z^\zeta(t)$ and $\psi = \phi(z^\zeta(t))$.

Suppose, for a proof by contradiction that there is $i \in \mathcal{I}$, a sequence of fluid model solutions $\{\zeta^n\}_{n \in \mathbb{N}}$ with $\zeta^n(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, and an associated sequence of times $\{t_n\}_{n \in \mathbb{N}}$, such that $z_i^n(t_n) \neq 0$ and $\left\{ \frac{z_i^n(t_n)}{\phi_i(z^n(t_n))} \right\}_{n \in \mathbb{N}}$ is unbounded. Here we use $z^n(\cdot)$ to represent $z^{\zeta^n}(\cdot)$ for simplicity. (Note also that ζ^n here is not the smoothed version of ζ used in Section 6.) Since $|\mathcal{J}| = \mathbf{J}$ is finite, R is a matrix of zeros and ones, and κ_i and α are fixed positive constants, by (8.2), we have that there exists $\{j_n^*\}_{n \in \mathbb{N}} \subset \mathcal{J}$ such that $R_{j_n^* i} = 1$ for each n and such that $\{p_{j_n^*}^{\zeta^n}(t_n)\}_{n \in \mathbb{N}}$ is an unbounded sequence of positive real numbers. By (3.17), for each n, since $p_{j_n^*}^{\zeta^n}(t_n) > 0$, we have

$$\sum_{k \in \mathcal{I}_+(z^n(t_n))} R_{j_n^* k} \phi_k(z^n(t_n)) = C_{j_n^*}. \tag{8.3}$$

Let $C_{min} = \min\{C_j : j \in \mathcal{J}\}$ and $\delta = \frac{C_{min}}{2\mathbf{I}} > 0$. Then for each n, there is $i_n^* \in \mathcal{I}_+(z^n(t_n))$ such that $R_{j_n^* i_n^*} = 1$ and $\phi_{i_n^*}(z^n(t_n)) > \delta$. Combining this with Lemma 7.4, we have

$$\frac{z_{i_n^*}^n(t_n)}{\phi_{i_n^*}(z^n(t_n))} < \frac{\widetilde{B}_\upsilon}{\delta} \tag{8.4}$$

for each n. Now, by (3.19) with i_n^* in place of i, we have

$$\frac{z_{i_n^*}^n(t_n)}{\phi_{i_n^*}(z^n(t_n))} = \left(\frac{\sum_{j \in \mathcal{J}} p_j^{\zeta^n}(t_n) R_{j i_n^*}}{\kappa_{i_n^*}} \right)^{\frac{1}{\alpha}}. \tag{8.5}$$

Since $R_{j_n^* i_n^*} = 1$ and $\{p_{j_n^*}^{\zeta^n}(t_n)\}_{n \in \mathbb{N}}$ is unbounded, it follows that $\frac{z_{i_n^*}^n(t_n)}{\phi_{i_n^*}(z^n(t_n))}$ diverges as $n \to \infty$, which contradicts (8.4). Because of this contradiction, it follows that M_υ is finite. ◻

With Lemma 8.1, we can prove the following strengthened form of Lemma 3.1, under the added assumption that the fluid model is critical, i.e., Assumption 1 holds.

Lemma 8.2. *Suppose Assumptions 1 and 2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, we have $\zeta(t) \in \mathbf{K}_{\upsilon^*}^{\mathbf{I}}$ for all $t \geq 0$, where*

$$\upsilon^* = \upsilon + M_\upsilon \max_{i \in \mathcal{I}} \rho_i. \tag{8.6}$$

Proof. Fix $i \in \mathcal{I}$ and a fluid model solution ζ with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$. For any $t \geq 0$, if $\zeta_i(t) = 0$, then the result holds for any $\upsilon^* > 0$. If $\zeta_i(t) \neq 0$, let $t_0 = \sup\{0 \leq s < t : \zeta_i(s) = 0\}$, where $\sup(\emptyset) = 0$. Then $\zeta_i(\cdot)$ is nonzero on $(t_0, t]$ and $\zeta_i(t_0) = 0$ if $t_0 > 0$. For $s \in (t_0, t]$ and $x \in [0, \infty)$, by (2.15),

$$\begin{aligned}
\overline{M}^i_t(x) &= \overline{M}^i_s(x + S^i_{s,t}) + \nu_i \int_s^t \overline{N}_i(x + S^i_{u,t})du \\
&\leq \overline{M}^i_s(x) + \int_s^t \nu_i \overline{N}_i(x + S^i_{u,t}) \frac{z_i(u)}{\Lambda_i(u)} \frac{\Lambda_i(u)}{z_i(u)} du \\
&\leq \overline{M}^i_s(x) + M_\upsilon \nu_i \int_s^t \overline{N}_i(x + S^i_{u,t}) \frac{d(-S^i_{u,t})}{du} du \\
&= \overline{M}^i_s(x) + M_\upsilon \nu_i \int_x^{x+S^i_{s,t}} \overline{N}_i(y)dy \quad \text{with } y = x + S^i_{u,t} \qquad (8.7) \\
&= \overline{M}^i_s(x) + M_\upsilon \nu_i \mu_i^{-1} \big(\overline{N}^e_i(x) - \overline{N}^e_i(x + S^i_{s,t})\big) \\
&\leq \overline{M}^i_s(x) + M_\upsilon \rho_i \overline{N}^e_i(x), \qquad (8.8)
\end{aligned}$$

where we used Lemma 8.1 for the second inequality. Now let $s \downarrow t_0$ in (8.8) to obtain

$$\overline{M}^i_t(x) \leq \overline{M}^i_{t_0}(x) + M_\upsilon \rho_i \overline{N}^e_i(x),$$

where $\overline{M}^i_{t_0}(x) \leq z_i(t_0) = 0$ if $t_0 > 0$ and $\overline{M}^i_{t_0}(x) = \overline{M}^i_0(x) \leq \upsilon \overline{N}^e_i(x)$ if $t_0 = 0$. Then for all $t \geq 0$, $i \in \mathcal{I}$,

$$\overline{M}^i_t(x) \leq \upsilon^* \overline{N}^e_i(x) \text{ for all } x \in [0, \infty), \qquad (8.9)$$

where υ^* is given by (8.6). Combining with Proposition 2.2 yields the desired result. □

Remark 8.1. The substitution step in (8.7) is similar to one used in the proof of Corollary 5.1 in [23]. However, the new crucial step here is to use the uniform bound on $\frac{z_i(\cdot)}{\Lambda_i(\cdot)}$ from Lemma 8.1.

8.3 Proofs of Theorems 5.2 and 5.3

Our proofs of Theorems 5.2 and 5.3 draw on some arguments in the proofs of Theorems 3.2 and 3.1, respectively, given in [23] for the case of a single class processor sharing queue. However, multiple details are more complicated in our more general setting. In particular, our Lyapunov function G is different, our fluid model solutions can have components that reach zero and we also have a less restrictive precompact set $\mathbf{K}^{\mathbf{I}}_\upsilon$ than in [23].

Proof of Theorem 5.2 (Monotone convergence of $\mathcal{G}^\zeta(\cdot)$ to zero). Fix $\upsilon > 0$. The monotonic decreasing property is an immediate consequence of Theorem 5.1. So it suffices to prove the uniform convergence to zero. Note that by Lemma 8.2, for υ^* as given there, and all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_\upsilon$, we have $\zeta(t) \in \mathbf{K}^{\mathbf{I}}_{\upsilon^*}$ for all $t \geq 0$. Given $\epsilon > 0$, let

$$\mathbf{G}_\epsilon = \{\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon^*} : G(\xi) < \epsilon\}. \qquad (8.10)$$

It suffices to show that there exists $T_\epsilon > 0$ such that for all ζ with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}_\upsilon$, we have $\zeta(t) \in \mathbf{G}_\epsilon$ for all $t \geq T_\epsilon$.

By Lemma 4.1, G is continuous on $\mathbf{M}^{\mathbf{I}}_{\upsilon^*}$. Then

$$\mathbf{G}^c_\epsilon = \mathbf{M}^{\mathbf{I}}_{\upsilon^*} \backslash \mathbf{G}_\epsilon = \{\xi \in \mathbf{M}^{\mathbf{I}}_{\upsilon^*} : G(\xi) \geq \epsilon\} \qquad (8.11)$$

is a closed set in the compact set $\mathbf{M}^{\mathbf{I}}_{\upsilon^*}$ and hence is compact. By Lemma 4.1(ii), we have $\mathbf{G}^c_\epsilon \cap \mathcal{M}^* = \emptyset$. Then by Lemma 3.5, $K(\xi) < 0$ for all $\xi \in \mathbf{G}^c_\epsilon$. By Lemma 3.4, K is upper semicontinuous on the compact set $\mathbf{G}^c_\epsilon$, and so it achieves its maximum there, which will be strictly negative. Let $\delta > 0$ be such that $K(\xi) \leq -\delta$

for all $\xi \in \mathbf{G}_\epsilon^c$. Then for any $t \geq 0$ and fluid model solution ζ with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, since $\underline{F}(\cdot) \geq 0$ and using Theorem 3.1, we have for any $t \geq 0$,

$$\begin{aligned} 0 \leq \mathcal{G}^\zeta(t) &= \mathcal{H}^\zeta(t) - \underline{F}\left(\widetilde{w}(\zeta(t))\right) \\ &\leq \mathcal{H}^\zeta(t) \\ &= \mathcal{H}^\zeta(0) + \int_0^t \mathcal{K}^\zeta(s)ds. \end{aligned} \tag{8.12}$$

Let $\tau_\epsilon^\zeta = \inf\{t \geq 0 : \zeta(t) \in \mathbf{G}_\epsilon\}$. Then by (8.12), since $\mathcal{K}^\zeta(s) = K(\zeta(s))$ where K has a maximum of $-\delta$ on $\mathbf{G}_\epsilon^c$, we have

$$\tau_\epsilon^\zeta \leq \frac{\mathcal{H}^\zeta(0)}{\delta} \leq \frac{1}{\delta(\alpha+1)} \sum_{i \in \mathcal{I}} \frac{\kappa_i \langle \chi, \vartheta_i^e \rangle (\upsilon^*)^{\alpha+1}}{\rho_i^\alpha} := T_\epsilon.$$

Since $t \to \mathcal{G}^\zeta(t)$ is non-increasing, by Theorem 5.1, it follows that $\zeta(t) \in \mathbf{G}_\epsilon$ for all $t \geq T_\epsilon$. Since T_ϵ does not depend on the particular ζ chosen, the desired result follows. □

Before proving Theorem 5.3, we first prove the following two lemmas. The first lemma is like Theorem 5.3, but with $\mathcal{M}^*$ in place of $\mathcal{M}$. The second lemma will be used to derive Theorem 5.3 from the first lemma.

Lemma 8.3. *Suppose that Assumptions 1 and 2 hold. Fix $\upsilon > 0$. For any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, $\zeta(t)$ converges towards $\mathcal{M}^*$ as $t \to \infty$, uniformly for all initial measures in $\mathbf{K}_\upsilon^{\mathbf{I}}$, i.e.,*

$$\lim_{t\to\infty} \sup\{\mathbf{d}_{\mathbf{I}}(\zeta(t), \mathcal{M}^*) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}\} = 0. \tag{8.13}$$

Furthermore, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\sup_{t\geq 0}\{\mathbf{d}_{\mathbf{I}}(\zeta(t), \mathcal{M}^*) : \zeta \text{ is a fluid model solution with } \zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}} \text{ and } \mathbf{d}_{\mathbf{I}}(\zeta(0), \mathcal{M}^*) < \delta\} \leq \epsilon. \tag{8.14}$$

Proof. Fix $\upsilon > 0$. By Lemma 8.2, with υ^* as given there, for any fluid model solution with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, we have $\zeta(t) \in \mathbf{K}_{\upsilon^*}^{\mathbf{I}}$ for all $t \geq 0$. For each $a > 0$, let

$$\mathbf{D}_a := \{\xi \in \mathbf{M}_{\upsilon^*}^{\mathbf{I}} : \mathbf{d}_{\mathbf{I}}(\xi, \mathcal{M}^*) \geq a\} \quad \text{and} \quad \mathbf{G}_a := \{\xi \in \mathbf{M}_{\upsilon^*}^{\mathbf{I}} : G(\xi) < a\}.$$

For the proof of (8.13), consider $\epsilon > 0$ fixed. Since $\xi \to d_{\mathbf{I}}(\xi, \mathcal{M}^*)$ is a continuous function on $\mathbf{M}_{\upsilon^*}^{\mathbf{I}}$, $\mathbf{D}_\epsilon$ is a closed subset of the compact set $\mathbf{M}_{\upsilon^*}^{\mathbf{I}}$ and hence is compact. By Lemma 4.1, G is strictly positive on $\mathbf{D}_\epsilon$. Then by the compactness of $\mathbf{D}_\epsilon$, there is $\delta_1(\epsilon) > 0$ such that $G(\xi) \geq \delta_1(\epsilon)$ for all $\xi \in \mathbf{D}_\epsilon$. Hence $\mathbf{D}_\epsilon \subset \mathbf{G}_{\delta_1(\epsilon)}^c = \mathbf{M}_{\upsilon^*}^{\mathbf{I}} \backslash \mathbf{G}_{\delta_1(\epsilon)}$ and so $\mathbf{G}_{\delta_1(\epsilon)} \subset \mathbf{D}_\epsilon^c = \mathbf{M}_{\upsilon^*}^{\mathbf{I}} \backslash \mathbf{D}_\epsilon$. By Theorem 5.2, there is $T_{\delta_1(\epsilon)} < \infty$ such that $\zeta(t) \in \mathbf{G}_{\delta_1(\epsilon)}$ for all $t \geq T_{\delta_1(\epsilon)}$, for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$. It follows that $d_{\mathbf{I}}(\zeta(t), \mathcal{M}^*) < \epsilon$ for all $t \geq T_{\delta_1(\epsilon)}$ and all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$. The result (8.13) follows since $\epsilon > 0$ was arbitrary.

For the proof of (8.14), fix $\epsilon > 0$ and let $\delta_1(\epsilon)$ be as defined above. Since G is a continuous function on the compact set $\mathbf{M}_{\upsilon+1}^{\mathbf{I}}$, it is uniformly continuous there. Also, G is zero on $\mathcal{M}_{\upsilon+1}^* = \mathcal{M}^* \cap \mathbf{M}_{\upsilon+1}^{\mathbf{I}}$. It follows that there is $\delta \in (0,1)$ such that $G(\xi) < \delta_1(\epsilon)$ whenever $\xi \in \mathbf{M}_{\upsilon+1}^{\mathbf{I}}$ and $\mathbf{d}_{\mathbf{I}}(\xi, \mathcal{M}_{\upsilon+1}^*) < \delta$. If ζ is a fluid model solution with $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ satisfying $\mathbf{d}_{\mathbf{I}}(\zeta(0), \mathcal{M}^*) < \delta$, then there is $\eta \in \mathcal{M}^*$ such that $\mathbf{d}_{\mathbf{I}}(\zeta(0), \eta) < \delta$. By the form of the elements of $\mathcal{M}^*$, we have for all $i \in \mathcal{I}$, $\langle \mathbb{1}_{[x,\infty)}, \eta_i \rangle \leq \langle \mathbb{1}, \eta_i \rangle \langle \mathbb{1}_{[x,\infty)}, \vartheta_i^e \rangle$ where $\langle \mathbb{1}, \eta_i \rangle \leq \langle \mathbb{1}, \zeta_i(0) \rangle + \delta \leq \upsilon + 1$, and so $\eta \in \mathbf{M}_{\upsilon+1}^{\mathbf{I}}$. It follows that $\mathbf{d}_{\mathbf{I}}(\zeta(0), \mathcal{M}_{\upsilon+1}^*) < \delta$ and hence, by the choice of δ, $G(\zeta(0)) < \delta_1(\epsilon)$. By Theorem 5.1, $t \to \mathcal{G}(t) = G(\zeta(t))$ is a non-increasing function and so $G(\zeta(t)) < \delta_1(\epsilon)$ for all $t \geq 0$. By Lemma 8.2, we also have that $\zeta(t) \in \mathbf{K}_{\upsilon^*}^{\mathbf{I}}$. Thus, $\zeta(t) \in \mathbf{G}_{\delta_1(\epsilon)} \subset \mathbf{M}_{\upsilon^*}^{\mathbf{I}} \setminus \mathbf{D}_\epsilon$, from the first part of this proof. It follows that $\mathbf{d}_{\mathbf{I}}(\zeta(t), \mathcal{M}^*) < \epsilon$ for all $t \geq 0$. The desired result (8.14) follows. □

The following lemma is a vector measure analogue of Lemma 4.4 in [23].

Lemma 8.4. *Suppose that $\xi \in \mathbf{M}^{\mathbf{I}}$ and $\theta > 0$ such that $\mathbf{d}_{\mathbf{I}}(\xi, \mathcal{M}^*) < \theta$. Then*

$$\mathbf{d}_{\mathbf{I}}(\xi, \mathcal{M}) \leq \max_{i \in \mathcal{I}} \xi_i([0, \theta)) + 2\theta. \tag{8.15}$$

Proof. There is $\eta \in \mathcal{M}^*$ such that $\mathbf{d_I}(\xi,\eta) < \theta$, and there is $a \in \mathbb{R}_+^{\mathbf{I}}$ and $b \in \mathcal{P}$ such that for each $i \in \mathcal{I}$, $\eta_i = a_i\delta_0 + b_i\vartheta_i^e$. Let $\vartheta_i^{e,b} = b_i\vartheta_i^e$ for $i \in \mathcal{I}$. Note that $\vartheta^{e,b} \in \mathcal{M}$. Then,

$$\mathbf{d_I}(\xi,\mathcal{M}) \le \mathbf{d_I}(\xi,\vartheta^{e,b}) \le \mathbf{d_I}(\xi,\eta) + \mathbf{d_I}(\eta,\vartheta^{e,b}) \le \theta + \max_{i\in\mathcal{I}} a_i, \tag{8.16}$$

where by the definition of the metric $\mathbf{d_I}$,

$$a_i = \eta_i(\{0\}) \le \xi_i([0,\theta)) + \theta \quad \text{for each } i \in \mathcal{I}. \tag{8.17}$$

Combining the above, yields the desired result (8.15). □

Proof of Theorem 5.3 (Convergence to the invariant manifold). We first note that for any $\theta > 0$ and any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, for $t \ge 0$, $i \in \mathcal{I}$, and $t_i = \sup\{s \le t : z_i(s) = 0\}$ where $z_i(s) = \langle \mathbb{1}, \zeta_i(s)\rangle$, using the fact that $\zeta_i(t)$ has no atom at $\{0\}$ and letting $s \downarrow t_i$ in (2.15) (when $t_i \ne t$), we have that

$$\begin{aligned}
\zeta_i(t)([0,\theta)) &= \overline{M}_t^i(0) - \overline{M}_t^i(\theta) \\
&= \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i \int_{t_i}^t (\overline{N}_i(S_{u,t}^i) - \overline{N}_i(\theta + S_{u,t}^i))du \\
&\le \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i M_\upsilon \int_0^{S_{t_i,t}^i} (\overline{N}_i(y) - \overline{N}_i(\theta + y))dy \\
&\le \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i M_\upsilon \int_0^{\theta} \overline{N}_i(y)dy \\
&\le \mathbb{1}_{\{t_i=0\}}\left(\overline{M}_0^i(S_{0,t}^i) - \overline{M}_0^i(\theta + S_{0,t}^i)\right) + \nu_i M_\upsilon \theta,
\end{aligned} \tag{8.18}$$

where for the third inequality, we used the change of variable $y = S_{u,t}^i$ and the upper bound of M_υ on $z_i(u)/\Lambda_i(u)$ for $u \in (t_i,t)$ afforded by Lemma 8.1, and for the last inequality we used the fact that $\overline{N}_i(\cdot)$ is bounded by one.

We first prove (5.2). For this, let $\epsilon > 0$ and

$$\theta = \frac{\epsilon}{3(1+\upsilon+M_\upsilon \max_{i\in\mathcal{I}}\nu_i)} \in \left(0,\frac{\epsilon}{3}\right).$$

By Lemma 8.3, there is $T_\theta^{(1)} > 0$ such that

$$\mathbf{d_I}(\zeta(t),\mathcal{M}^*) < \theta \quad \text{for all } t \ge T_\theta^{(1)}, \tag{8.19}$$

for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$. Then for each such fluid model solution, by Lemma 8.4, we have

$$\mathbf{d_I}(\zeta(t),\mathcal{M}) \le \max_{i\in\mathcal{I}} \zeta_i(t)([0,\theta)) + 2\theta \quad \text{for all } \ t \ge T_\theta^{(1)}, \tag{8.20}$$

and by (8.18), the fact that $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, and since by Lemma 8.1,

$$S_{0,t}^i = \int_0^t \frac{\phi_i(z(u))}{z_i(u)}du \ge \frac{t}{M_\upsilon} \quad \text{when } t_i = 0,$$

we have for each $i \in \mathcal{I}$,

$$\begin{aligned}
\zeta_i(t)([0,\theta)) &\le \mathbb{1}_{\{t_i=0\}}\overline{M}_0^i(S_{0,t}^i) + \nu_i M_\upsilon\theta \\
&\le \upsilon\overline{N}_i^e(t/M_\upsilon) + \nu_i M_\upsilon \theta.
\end{aligned} \tag{8.21}$$

Let $T_\theta^{(2)}$ be such that for each $i \in \mathcal{I}$, $\overline{N}_i^e(t/M_\upsilon) < \theta$ for all $t \ge T_\theta^{(2)}$. Combining this with (8.20), (8.21), and the definition of θ, we see that for all $t \ge T_\theta^{(1)} \vee T_\theta^{(2)}$, for any fluid model solution ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$, we have

$$\mathbf{d_I}(\zeta(t),\mathcal{M}) \le \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon. \tag{8.22}$$

Since $\epsilon > 0$ was arbitrary, it follows that (5.2) holds.

We now turn to proving (5.3). For this, fix $\epsilon \in (0,1)$. It suffices to consider such an ϵ, since a δ that works for such an ϵ also works for all larger ϵ. Because $\overline{N}_i^e(\cdot)$ is uniformly continuous on $[0,\infty)$ and $i \in \mathcal{I}$ takes finitely many values, there is $h_\epsilon > 0$ such that for all $i \in \mathcal{I}$ and $0 \le h \le h_\epsilon$, we have

$$\sup_{x\in[0,\infty)} (\overline{N}_i^e(x) - \overline{N}_i^e(x+h)) < \frac{\epsilon}{4(\upsilon+1)}. \tag{8.23}$$

Let

$$\theta = \min\left(\frac{h_\epsilon}{3}, \frac{\epsilon}{4(1+M_\upsilon \max_{i\in\mathcal{I}} \nu_i)}\right). \tag{8.24}$$

By the last part of Lemma 8.3, with θ in place of ϵ there, we can find $\delta \in (0, \theta \wedge 1)$ (not depending on ζ) such that

$$\mathbf{d_I}(\zeta(t), \mathcal{M}^*) \le \theta \quad \text{for all } t \ge 0, \tag{8.25}$$

for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ and $\mathbf{d_I}(\zeta(0), \mathcal{M}) < \delta$. It follows from Lemma 8.4 that for all such fluid model solutions ζ,

$$\mathbf{d_I}(\zeta(t), \mathcal{M}) \le \max_{i\in\mathcal{I}} \zeta_i(t)([0,\theta)) + 2\theta \quad \text{for all } t \ge 0. \tag{8.26}$$

Since $\mathbf{d_I}(\zeta(0), \mathcal{M}) < \delta$, there is $b \in \mathcal{P}$ such that $\mathbf{d_I}(\zeta(0), \vartheta^{e,b}) < \delta$ where $\vartheta_i^{e,b} = b_i \vartheta_i^e$ and $b_i \le \upsilon + \delta$ for each $i \in \mathcal{I}$. It follows from this and (8.18) that for any $t \ge 0$ and $i \in \mathcal{I}$,

$$\begin{aligned}
\zeta_i(t)([0,\theta)) &\le \mathbb{1}_{\{t_i=0\}}(\langle \mathbb{1}_{(S_{0,t}^i, \theta+S_{0,t}^i]}, \zeta_i(0)\rangle) + \nu_i M_\upsilon \theta \\
&\le \mathbb{1}_{\{t_i=0\}}(b_i \langle \mathbb{1}_{((S_{0,t}^i-\delta)^+, \theta+S_{0,t}^i+\delta)}, \vartheta_i^e\rangle + \delta) + \nu_i M_\upsilon \theta \\
&= \mathbb{1}_{\{t_i=0\}}(b_i(\overline{N}_i^e((S_{0,t}^i-\delta)^+) - \overline{N}_i^e(\theta + S_{0,t}^i + \delta)) + \delta) + \nu_i M_\upsilon \theta \\
&\le (\upsilon+\delta)(\sup_{x\in[0,\infty)} (\overline{N}_i^e(x) - \overline{N}_i^e(x+\theta+2\delta)) + \delta + \nu_i M_\upsilon \theta \\
&\le (\upsilon+1)\frac{\epsilon}{4(\upsilon+1)} + \theta + \nu_i M_\upsilon \theta \\
&\le \frac{\epsilon}{2},
\end{aligned} \tag{8.27}$$

where we used (8.23), the facts that $\delta < \theta \wedge 1$ and $\theta + 2\delta \le 3\theta \le h_\epsilon$ for the second last inequality, and we used the definition of θ for the last inequality. Combining (8.26) with (8.27) and the fact that $\theta \le \frac{\epsilon}{4}$, we find that

$$\mathbf{d_I}(\zeta(t), \mathcal{M}) \le \epsilon \quad \text{for all } t \ge 0, \tag{8.28}$$

for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_\upsilon^{\mathbf{I}}$ and $\mathbf{d_I}(\zeta(0), \mathcal{M}) < \delta$. Hence (5.3) holds. □

Appendix A Hazard Rate

Definition A.1. *Assume that ξ is a probability measure on $\mathbb{R}_+$ defining the distribution of an absolutely continuous, non-negative random variable with probability density function $j(\cdot)$ and cumulative distribution function $J(\cdot)$. The hazard rate function for ξ is defined by*

$$q(x) = \frac{j(x)}{1-J(x)} \quad \textit{for} \ \ 0 < x < x^*,$$

where $x^* = \inf\{x \ge 0 : J(x) = 1\}$. The distribution is said to have bounded hazard rate if there is a finite constant L such that

$$q(x) \le L \quad \textit{for all } 0 < x < x^*.$$

It turns out that in order to have a bounded hazard rate, the support of the distribution must be unbounded and so, in this case, $x^* = \infty$ and

$$q(x) \leq L \quad \text{for all } x \in (0, \infty).$$

Under Assumption 2, that ϑ_i has bounded hazard rate for all $i \in \mathcal{I}$, we have also that $\frac{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle}$ is bounded for all $x \in [0, \infty)$ and $i \in \mathcal{I}$. To see this, note that if q_i, j_i, J_i are the hazard rate, probability density and cumulative distribution function, respectively, for ϑ_i for some $i \in \mathcal{I}$ and L_i is a bound for q_i, then for all $x \geq 0$, we have

$$\int_x^\infty \big(1 - J_i(y)\big) dy \geq \frac{1}{L_i} \int_x^\infty j_i(y) dy = \frac{1 - J_i(x)}{L_i},$$

and consequently,

$$\frac{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle}{\langle \mathbb{1}_{(x,\infty)}, \vartheta_i^e \rangle} = \frac{1 - J_i(x)}{\mu_i \int_x^\infty \big(1 - J_i(y)\big) dy} \leq \frac{L_i}{\mu_i}.$$

We now give some examples of common distributions with bounded hazard rates.

- *Gamma Distribution.* The probability density function has the form

$$j(x) = \frac{a^b x^{b-1}}{\Gamma(b)} e^{-ax} \quad \text{for } x > 0,$$

  where $a, b > 0$ are parameters. The hazard rate corresponding to this Gamma distribution is

$$q(x) = \frac{x^{b-1} e^{-ax}}{\int_x^\infty y^{b-1} e^{-ay} dy} \quad \text{for } x > 0.$$

  If $0 < b < 1$, the hazard rate function is decreasing, but it is unbounded on $(0, \infty)$. If $b = 1$, the Gamma distribution is the exponential distribution with constant hazard rate function equal to a, which is clearly bounded. If $b > 1$, the hazard rate function is increasing and using the asymptotic behavior of the incomplete gamma function at infinity, we see that $\lim_{x \to \infty} q(x) = a$ and so q is bounded.

- *Pareto Distribution.* The probability density function has form

$$j(x) = \frac{a x_m^a}{x^{a+1}} \quad \text{for } x \geq x_m,$$

  where $a > 0$ and $x_m > 0$ are parameters. The corresponding hazard rate function is $q(x) = \frac{a}{x}$ for $x \geq x_m$. This function is decreasing and bounded on $[x_m, \infty)$. Note that we must have $a > 2$ in order for such a distribution to have finite first and second moments.

- *Lognormal Distribution.* A random variable X follows the lognormal distribution if $Y = \log(X)$ is normally distributed. The probability density function is therefore given by

$$j(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - a)^2}{2\sigma^2} \right) \quad \text{for } x > 0,$$

  where $a \in (0, \infty)$ and $\sigma > 0$ are parameters. The hazard rate function is given by

$$q(x) = \frac{\exp\Big(-\frac{(\ln x - a)^2}{2\sigma^2} \Big)}{x\sigma\sqrt{2\pi}\Big(1 - \Phi\big(\frac{\ln x - a}{\sigma}\big)\Big)}, \quad \text{for } x > 0.$$

  where Φ is the cumulative distribution function for the standard normal distribution. It can be shown, see e.g., Sweet [24], that the hazard rate function tends to zero at zero and infinity and has a maximum in between. Thus it has an inverted bathtub shape and is bounded.

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